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ABSTRACT

Every reasoning task requires the gathering of information. This process is costly, and therefore, to solve a problem efficiently, one should have a method of distinguishing relevant facts from irrelevant facts. Otherwise a system would spend precious time processing facts which are irrelevant to the task at hand.

This paper analyses a formalization of relevance called *dependency models* ([Pearl 1986a]). These models consist of a finite collection of propositions U and a three place predicate I over disjoint subsets of these propositions. The *statement* $I(x, z, y)$ is assigned a truth value if the propositions in x are *irrelevant* to the propositions in y, once z is known.

In particular we examine the limitation of representing the truth assignment of I by Directed Acyclic Graphs (DAGs). We prove that these graphical models cannot be characterized as *Horn-axioms*, and conjecture that neither are *disjunctive axioms* powerful enough for this goal. This conjecture is proven for a subclass of disjunctive axioms called *functional-restricted axioms*. Finally, we examine dependency models comprising families of graphs, directed as well as undirected. We find a *complete axiomatization* for these multi-graph models which consists of three axioms: Symmetry, Decomposition and Weak-union. We propose these models as a scheme for representing *semi-graphoids* and *graphoids*.

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Table of Contents

1. INTRODUCTION

Every reasoning task, whether as simple as scheduling your next day appointments, or as complex as planning a space voyage, requires the gathering and learning of information. The learning process however, is costly in time and, hence, to solve a problem efficiently, one should have a clear idea of which facts are *relevant* and worth acquiring. This might be the main characteristic of expert's specialization; The skill to identify the relevant facts and learn these facts, rather than having an immediate answer for every query.

The ability to distinguish relevant facts from irrelevant facts is also important in any ***reasoning engine*** that deals with ***knowledge and beliefs***. Otherwise this engine would spend precious time processing facts which are irrelevant to the task at hand. For example, knowing the grade point average (GPA) of student A, usually tells us nothing about the GPA of student B. However, once we learn that the two students are taking the same courses and always study together for exams, their GPAs become related. This change of dependencies is called *nonmonotonic*, since propositions which were irrelevant before, become relevant when new information is learned. On the other hand, a *monotonic* change of dependencies occurs when propositions that were relevant before become irrelevant. For example, deciding whether one will go swimming in the ocean depends on the expected temperatures. However, once one hears on the radio that a tornado has swept the shore, no longer are the temperatures relevant to one's decision.

Dependency models are formal ways of representing such *informational dependencies*. A ***dependency model M*** is a list of triplets (x, z, y) for which the propositions represented by the set x are irrelevant to the propositions represented by the set y, once we know z. Equivalently, a dependency model can be regarded as a truth assignment rule for the predicate $I(x, z, y)_M$, where I stands for "x is irrelevant to y once z is known".

Clearly, to keep an explicit list of all triplets is impractical, due to the enormous number of combinations of events. Moreover the list changes in time. As new information is learned, triplets might be added or deleted, making the update of the list an insurmountable task. Therefore, instead of an explicit list, an efficient mechanism of assigning truth values to $I(x, z, y)_M$ is necessary.

The device of *conditional independence* in probability theory is an example for such a mechanism ([Pearl 1986a]). Associating the notion of *irrelevancy* to *conditional independence* suggests that we define $I(x, z, y)_P$ to be true (i.e. irrelevant) iff the variables x and y are conditionally independent given z, in some probability distribution P. i.e.,

$$P(x,y \mid z) = P(x \mid z) \cdot P(y \mid z)$$

This definition captures our intuition about how dependencies are changed when learning new facts. Two independent variables which become dependent upon learning a new fact, exhibit a nonmonotonic behavior, while variables that were originally dependent and become independent present a monotonic change. Thus, probability theory has the expressive power to represent the changes that occur when learning new facts and therefore, as suggested in [Pearl & Verma 1986], "probability theory can provide the machinery for identifying which propositions are relevant to each other with any given state of knowledge".

Relational database gives rise to another machinery, *Embedded Multivalued Dependencies* (EMVD, [Fagin 1977]), which is also capable of capturing the dynamics of informational dependencies. An Embedded Multivalued Dependency is a restriction defined on a collection of tuples R called a *database*. Each tuple is an unordered list of attributes, or sets of attributes, whose values belong each to its own domain. For example, attributes of a Personnel database might be: name, age and phone number. A sample tuple of this database might be $< Dan\ Geiger, 29, 825\text{–}6010 >$. Conventionally, $<x, y, z>$ denotes a tuple over the attributes x, y and z. Let x_1, x_2, y_1, y_2 and z_1 be values from the domains of three disjoint sets of attributes x, y and z respectively, then an Embedded Multivalued Dependency $I(x, z, y)_{EMVD}$ holds iff

$$<x_1, y_1, z_1> \& <x_2, y_2, z_1> \rightarrow <x_1, y_2, z_1>$$

In other words, this restriction means that if the two tuples $<x_1, y_1, z_1>$ and $<x_2, y_2, z_1>$ appear in a database then the tuple $<x_1, y_2, z_1>$ must appear as well. The idea behind this definition is that once the value of z is fixed, knowing the value of y cannot further restrict the permissible values of x. Hence, in this sense, EMVD captures the notion of irrelevancy between x and y. This definition was also used in [Shenoy & Shafer 1987] to devise a "qualitative" extension of probabilistic dependencies.

The use of similar syntactic notation (i.e. $I(x, z, y)$) does not assure us any semantic relation between the two classes of models. Nevertheless, the following properties are shared both by probabilistic models and by relational database EMVD models.

(Symmetry)

$$I(x, z, y) \Rightarrow I(y, z, x) \tag{1.a}$$

(Decomposition)

$$I(x, z, y \cup w) \Rightarrow I(x, z, y) \tag{1.b}$$

(Weak Union)

$$I(x, z, w \cup y) \Rightarrow I(x, z \cup w, y) \tag{1.c}$$

(Contraction)

$$I(x, z, y) \ \& \ I(x, z \cup y, w) \Rightarrow I(x, z, y \cup w) \tag{1.d}$$

Surprisingly, these common properties have an elegant interpretation that captures our basic demand from a reasoning system, *consistency*. By consistency we mean that acquiring irrelevant facts does not change the relevance status of other propositions in the system. Any information that was irrelevant remains irrelevant (1.c), and every fact that was relevant stays relevant (1.d). This interpretation is most easily seen by rewriting axioms (1.c) and (1.d) in the following way:

$$I(x, z, y) \Rightarrow \left[I(x, z, w \cup y) \Longleftrightarrow I(x, z \cup y, w) \right]$$

Hence, if (w) is irrelevant to (x) then it remains irrelevant after learning an irrelevant fact (y), and if (w) was relevant, it stays relevant.

A dependency model obeying axioms (1.a) through (1.d) is called a *semi-graphoid*. For example, probabilistic and EMVD models, can readily be shown to obey these axioms, hence they are semi-graphoids. The converse, namely, that for every semi-graphoid there exists an equivalent probabilistic model, has been conjectured in [Pearl 1986a]. This conjecture, called the completeness conjecture, is also equivalent (as shown in section 3) to the assertion that every property of *conditional independence* relation is logically implied by the semi-graphoid axioms.

This intuitively-appealing interpretation of semi-graphoid axioms motivated Pearl to propose these constraints as a formalization of the notion of informational dependencies. The completeness conjecture, if validated, would further imply that probabilistic conditional independence offers an equivalent formalization. Accordingly, two obvious ways to maintain a system that obeys these axioms follows. The first is to keep all triplets (x, z, y) for which $I(x, z, y)$ holds. The impracticality of this implementation was already discussed. The second is to employ either a probabilistic model or a database model because both models are guaranteed to obey axioms (1.a) through (1.d). The deficiency of these representations is twofold. First, to answer a query "Does $I(x, z, y)$ hold ?", requires, on the average, an exponential amount of work. Clearly, no reasoning system can allow such a cost, because the answer to this query is the preliminary tool that these systems should provide. Moreover, the ease and conviction by which people judge this query suggest that the answer can be obtained by a few primitive and quick operations on the representation scheme. Secondly, probabilistic models, as well as database models, involve derivational steps which do not match human reasoning, i.e., adding and multiplying many numbers. Therefore, even though a system based on probability might give "right" answers, it would be extremely difficult for it to give a meaningful interpretation as to how these conclusions were drawn.

A step towards resolving these deficiencies comes in the form of *dependency graphs*. Dependency graphs are Undirected Graphs (UG) or Directed Acyclic Graphs (DAGs) along with some criteria for determining the validity of a *statement* $I(x, z, y)$. The nodes in these graphs represent propositional variables. The arcs represent local dependencies among related propositions and the criterion could, for example, be whether the nodes in z separate the nodes in x from those in y. In dependency graphs the validity of a statement $I(x, z, y)$ can be determined in time proportional to the number of nodes in the graph. Hence the first problem of exponential calculations is resolved in these graphs.

The graph model also seems to be closer to the human reasoning process. The use of graph-related concepts (such as "threads of thoughts", "lines of reasoning", "connected ideas" etc) in everyday language suggests that derivational steps made on graphs are far easier to explain then any numerical calculation ([Pearl 1986a]).

These promising features of dependency graphs motivated the work reported in this thesis. The sections are organized as follows: Section 2 reviews dependency graphs in more detail. It is based on [Pearl 1986a], [Pearl & Paz 1986], [Pearl & Verma 1987] and should be viewed as a summary of these papers. Section 3 explores formal issues regarding the notion of completeness. Section 4 and 5 examine the restrictions imposed by the DAG model on the truth assignment of $I(x, z, y)$. We show that these restrictions cannot be formalized as Horn-axioms and therefore are too complex to be used as an inference mechanism. Section 6 suggests an alternative scheme for representing semi-graphoids. Finally, section 7 summarizes the results and outlines the relation between dependency models and relational databases.

2. DEPENDENCY GRAPHS

The topic of dependency models is best introduced by quoting excerpts from [Pearl 1986a], [Pearl & Paz 1986] and [Pearl & Verma 1987].

"Despite the prevailing use of graphs as metaphors for communicating and reasoning about dependencies, the task of capturing dependencies by graphs is not at all trivial. When we deal with a phenomenon where the notion of neighborhood or connectedness is explicit (e.g., family relations, electronic circuits, communication networks, etc.), we have no problem configuring a graph which represents the main features of the phenomenon. However, in modeling conceptual relations such as causation, association and relevance, it is often hard to distinguish direct neighbors from indirect neighbors; so, the task of constructing a graph representation then becomes more delicate. The notion of conditional independence in probability theory provides a perfect example of such a task. For a given probability distribution P and any three variables x, y, z, while it is fairly easy to verify whether knowing z renders x independent of y, P does not dictate which variables should be regarded as direct neighbors. In other words, we are given the means to test whether any given element z *intervenes* in a relation between elements x and y, but it remains up to us to configure a graph that encodes these interventions. We shall see that some useful properties of dependencies and relevancies cannot be represented graphically and the challenge remains to devise graphical schemes that minimize such deficiencies.

Ideally, we would like to represent dependency between elements by a path connecting their corresponding nodes in some graph G. Similarly, if the dependency between elements x and y is not direct and is mediated by a third element, z, we would like to display z as a node that intercepts the connection between x and y, i.e., z is a cutset separating x from y. This correspondence between conditional dependencies and cutset separation in undirected graphs forms the basis of the theory of Markov fields [Lauritzen 1982].

Definition: An *Undirected Graph Dependency* model (UGD) M_G is defined in terms of an undirected graph G. If X, Y and Z are three disjoint subsets of nodes in G then by definition $I(X, Z, Y)_G$ iff every path between nodes in X and Y contains at least one node in Z. In other words, Z is a cutset separating X from Y.

Ideally, we would like to require that if the removal of some subset S of nodes from the graph G renders nodes x and y disconnected (written $I(x, S, y)_G$), then this separation should correspond to conditional independence between x and y given S, namely,

$$I(x, S, y)_G \Rightarrow I(x, S, y)_M$$

and, conversely,

$$I(x, S, y)_M \Rightarrow I(x, S, y)_G$$

This would provide a clear graphical representation for the notion that x does not affect y directly, that its influence is mediated by the variables in S. Unfortunately, we shall next see that these two requirements are too strong; there is often no way of using vertex separation in a graph to display *all* dependencies and independencies embodied in some probabilistic models, even those portraying simple, everyday experiences.

Definition: An undirected graph G is a *dependency map* (D-map) of a dependency model M (over variables U) if there is a one-to-one correspondence between the elements of U and the nodes of G, such that for all disjoint subsets, x, y, z, of elements we have:

$$I(x, z, y)_M \Rightarrow I(x, z, y)_G$$

Similarly, G is an *Independency map* (I-map) of M if:

$$I(x, z, y)_M \Leftarrow I(x, z, y)_G$$

G said to be a *perfect map* of M if it is both a D-map and I-map.

A D-map guarantees that vertices found to be connected are, indeed, dependent; however, it may occasionally display dependent variables as separated vertices. An I-map works the opposite way: it guarantees that vertices found to be separated always correspond to genuinely independent variables but does not guarantee that all those shown to be connected are, in fact, dependent. Empty graphs are trivial D-maps, while complete graphs are trivial I-maps.

It is not hard to see that many reasonable models of dependency have no perfect maps. This occurs, for example, in models where $I(x, z, y)$ exhibits *nonmonotonic* behavior; totally unrelated propositions become relevant to each other upon learning new facts. A nonmonotonic model M, implying both $I(x, z_1, y)_M$ and $\neg I(x, z_1 \cup z_2, y)_M$ cannot have a graph representation which is both an I-map and a D-map, because graph separation always satisfies $I(x, z_1, y)_G \Rightarrow I(x, z_1 \cup z_2, y)_G$ for any two subsets z_1 and z_2 of vertices. Thus, D-mapness forces G to display z_1 as a cutset separating x and y, while I-mapness prevents $z_1 \cup z_2$ from separating x and y. No graph can satisfy these two requirements simultaneously.

This weakness in the expressive power of undirected graphs severely limits their ability to represent probabilistic dependencies. A simple example illustrating this point is an experiment with two coins and a bell that rings whenever the outcomes of the two coins are the same. If we ignore the bell, the coin outcomes, x and y, are mutually independent, i.e., $I(x, \varnothing, y)$. However, if we notice the bell (z), then learning the outcome of one coin should change our opinion about the other coin, namely, $\neg I(x, z, y)$.

How can we graphically represent these simple dependencies between the coins and the bell or, in general, between a set of multiple causes leading to a common consequence? If we take the naive approach and assign links to (z, x) and (z, y), leaving x and y unlinked, we get the graph x—z—y. This graph is not an I-map because it asserts that x and y are independent given z, which is wrong. If we add a link between x and y as well, we get the trivial I-map of a complete graph, which no longer alerts us to the obvious fact that the two coins are genuinely independent since the bell is merely a passive device which does not affect their interaction. Such dependencies, however, can be represented completely by using the richer language of directed graphs.

Definition: A *Directed Acyclic Graph Dependency* model (DAGD) M_D is defined in terms of a directed acyclic graph (DAG) D. If X, Y and Z are three disjoint subsets of nodes in D, then by definition $I(X, Z, Y)_D$ iff there is no bi-directed path from a node in X to a node in Y along which every node with converging arrows either is or has a descendent in z and every other node is outside Z.

The latter condition corresponds to ordinary cutset separation in undirected graphs while the former conveys the idea that the inputs of any causal mechanism become dependent once the output is known. This criterion was called *d-separation* in [Pearl 1986b]. In Figure 1, for example, $X = \{2\}$ and $Y = \{3\}$ are d-separated by $Z = \{1\}$ (i.e. $(2, 1, 3) \in M_D$) because knowing the common cause 1 renders its two possible consequences, 2 and 3, independent. However X and Y are not d-separated by $Z' = \{1, 5\}$ because learning the value of the consequence 5, renders its causes 2 and 3 dependent, like opening a pathway along the converging arrows at 4.

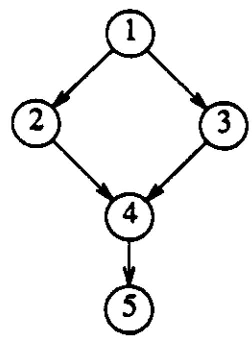

Figure 1. A DAG displaying *d*-separation: $(2, 1, 3) \in M_D$ while $(2, \{1, 5\}, 3) \notin M_D$

The introduction of directionality, enables us to perfectly represent the dependencies embodied in the bell and coins example. The fact that the sound of the bell is functionally determined by the outcomes of the two coins is represented by the network *coin 1* → *bell* ← *coin 2*, without connecting *coin 1* to *coin 2*. This pattern of converging arrows is interpreted as stating that the outcomes of the two coins are normally independent but may become dependent upon knowing the outcome of the bell (or any other external evidence bearing on that outcome)."

The success of capturing some non-monotonic dependencies still does not guarantee that DAGD models are capable of capturing all the dependencies of an arbitrary probabilistic model. In fact, it is easy to construct a probabilistic model that cannot be perfectly represented by a DAG. Before presenting such an example, we state the following axiom which is proven in section 4 to hold for every DAG:

$$I(A_2, A_1, A_3) \;\&\; I(A_3, A_2, A_4) \;\&\; I(A_4, A_3, A_1) \Rightarrow I(A_1, \emptyset, A_4).$$

The conditional independence relation in probability theory, however, is not constrained to obey this axiom and allows the co-existence of the following conditions:

1. $I(A_2, A_1, A_3)$ 2. $I(A_3, A_2, A_4)$ 3. $I(A_4, A_3, A_1)$ 4. $\neg I(A_1, \emptyset, A_4)$

For instance, the standard normal distribution over four variables with the following covariance matrix

$$\Gamma = (\rho_{i,j}) = \begin{bmatrix} 1 & \rho & \rho & \rho^4 \\ \rho & 1 & \rho^2 & \rho \\ \rho & \rho^2 & 1 & \rho^3 \\ \rho^4 & \rho & \rho^3 & 1 \end{bmatrix}$$

satisfies conditions (1) through (4). The covariance matrix is constructed as a possible solution to the following equations, each of which forces one of the four conditions (we assume that each A_i consists of one variable simply denoted i):

$$\rho_{2,3} = \rho_{1,2} \cdot \rho_{1,3}$$

$$\rho_{3,4} = \rho_{2,3} \cdot \rho_{2,4}$$

$$\rho_{1,4} = \rho_{1,3} \cdot \rho_{3,4}$$

$$\rho_{1,4} \neq 0$$

Any value of ρ such that $0 < \rho < 1/16$, would render Γ positive definite and therefore a valid covariance matrix.

This numerical example should be contrasted with the coins and bell example. While the latter is a cause and effect configuration, common in every-day reasoning, the former is merely an example validated by the formalism of probability theory. Indeed The DAGD model cannot perfectly represent an arbitrary semi-graphoid (or a probabilistic model) but it is capable of capturing many of the dependencies common in real-life reasoning.

Having realized that simple graphical representation are not powerful enough for perfectly representing semi-graphoids, researchers have aimed towards a less ambitious goal. Instead of a perfect map of semi-graphoid M they searched for approximations, i.e., an I-map that exhibits all the dependencies of M but not all of M's independencies. The natural requirement from these I-maps is that the number of undisplayed independencies be minimized.

The task of finding a DAG which is a minimal I-map of a given semi-graphoid M was solved in [Pearl & Verma 1987]. Their algorithm consists of the following steps: assign a total ordering to the variables of M. For each variable n of M, identify a minimal set of predecessors S_n that makes n independent of all its other predecessors (in the ordering of the first step). Assign a direct link from every variable in S_n to n. The resulting DAG is an I-map of M, and it is minimal in the sense that no edge can be deleted without destroying its I-mapness.

No equivalent algorithm is known to work for undirected graphs, unless the generating semi-graphoid model obeys the following additional axiom:

$$\textit{intersection} \quad I(x,\ z \cup y,\ w)\ \&\ I(x,\ z \cup w,\ y) \Rightarrow I(x,\ z,\ y \cup w). \tag{1.f}$$

This axiom, in addition to the semi-graphoid axioms (1.a through 1.d), define a class of dependency models called ***graphoids***. The name for this class is derived from the fact that these five axioms hold both in UGs and DAGs. The intersection axiom also holds for ***non-extreme*** probabilistic models, where the distributions are limited to be strictly positive. Without this limitation intersection would not hold. For instance, as argued in [Pearl 1986a], if y stands for the proposition "The water temperature is above 0^oC," and w stands for "The water temperature is above 32^oF," then, clearly, knowing the truth of either one of them renders the other superfluous. Yet, contrary to (1.f), this should not render both y and w irrelevant to a third proposition x, say, whether we will enjoy swimming in that water.

The ability to construct graphical I-maps (DAGs) for an arbitrary semi-graphoids is an additional superiority of DAGD models over UGD models. UGD models, so far, can be constructed as minimal I-maps only for graphoids ([Pearl & Paz 1986]). Thus, as pointed out in [Pearl & Verma 1987], the use of UGD models as a representation scheme rules out logical, functional and definitional constraints (which cannot be represented by non-extreme probabilistic models). This explains our focus on DAGD models as a possible scheme for approximating semi-graphoids.

3. COMPLETENESS

Dependency graphs, as shown in the previous section, are insufficient for perfectly representing an arbitrary semi-graphoid. The natural question to ask is which semi-graphoids lend themselves to a graphical representation or equivalently, which additional constraints should $I(x, z, y)$ obey in order to be perfectly represented by a UG or a DAG. This question is formalized in this section using the concept of *completeness*. We start by stating the completeness theorem for UGD models.

Definition: [Pearl & Paz 1986] A dependency model M is said to be a *graph-isomorph* if there exists a graph $G = (U, E)$ which is a perfect map of M, i.e., for every three disjoint subsets x, y and z of U, we have:

$$I(x, z, y)_M \iff I(x, z, y)_G$$

Theorem 1: [Pearl & Paz, 1986] A necessary and sufficient condition for a dependency model M to be graph-isomorph is that $I(x, z, y)_M$ satisfies the following five independent axioms (the subscript M dropped for clarity):

(symmetry)

$$I(x, z, y) \Rightarrow I(y, z, x) \tag{2.a}$$

(decomposition)

$$I(x, z, y \cup w) \Rightarrow I(x, z, y) \;\&\; I(x, z, w) \tag{2.b}$$

(intersection)

$$I(x, z \cup w, y) \;\&\; I(x, z \cup y, w) \Rightarrow I(x, z, y \cup w) \tag{2.c}$$

(strong union)

$$I(x, z, y) \Rightarrow I(x, z \cup w, y)$$

(transitivity) (2.d)

$$I(x, z, y) \Rightarrow I(x, z, \gamma) \ or \ I(\gamma, z, y)$$

(2.e)

Remark: Throughout this thesis we use the convention that in every instantiation of I the arguments are disjoint and that Greek letters indicate single variables. For example in (2.e) γ is a single element disjoint of $x \cup y \cup z$. We further assume that $I(x, z, \emptyset)$ always holds.

The proof of theorem 1 first verifies that the axioms in (2) are obeyed by vertex separation. Then it gives a constructive algorithm that produces a graph G which is a perfect map of an arbitrary dependency model M that obeys these axioms. The proof uses the fact that (2.c) and (2.d) imply the converse of (2.b) which makes I completely defined by the triplets (α, Z, β) where α and β are singletons. The construction of G is achieved by starting with a complete graph and simply deleting every edge (α, β) for which a triplet of the form (α, Z, β) appears in M. Using induction, the resulting graph is shown to be a perfect map of M.

Axioms (2.a)-(2.e) are said to be *complete* because these axioms delineate precisely the dependency models that are graph-isomorph, thus reflecting the *definitional* role of the axioms. A different definition of completeness is often used emphasizing the *derivational* role of axioms, i.e. the ability to infer new independency statements from a given set and be guaranteed that all statements that are *logically implied* by the set (i.e., true in all graphs that obey the initial set) are indeed *derivable* by successive application of the axioms. In all definitions that follow, we assume a fixed family of models M. For example M can be taken to be the set of graph-isomorph dependency models, or the set of all dependency models induced by probabilistic distributions.

Definition: An *axiom*

$$I(x_1,y_1,z_1) \,\&\, \cdots \,\&\, I(x_n,y_n,z_n) \Rightarrow I(\hat{x}_1,\hat{y}_1,\hat{z}_1) \; or \; \cdots \; or \; I(\hat{x}_m,\hat{y}_m,\hat{z}_m)$$

is *sound* for **M** if every model in **M** that obeys the antecedents of the axiom also obeys at least one of the statements of the disjunction on the right-hand-side of the implication. When $m=1$ the axiom is said to be a *Horn axiom.*

Definition: Let Σ be a set of statements and let σ be a single statement. Let **A** be a set of axioms. σ is *logically implied* by Σ iff every model in **M** that obeys Σ also obeys σ, i.e, there exists no "counterexample" model in **M** that satisfies Σ but does not satisfy σ. When σ is logically implied by Σ we use the notation: $\Sigma \models \sigma$, and say that σ is a *single consequence* of Σ. When σ can be derived from Σ by using the axioms in **A** we write: $\Sigma \models_A \sigma$, and say that σ is *derivable from* Σ.

Definition: A set of axioms is *weakly complete* if for every set of independency statements Σ and for every single statement σ we have:

$$\Sigma \models \sigma \text{ iff } \Sigma \models_A \sigma.$$

In other words every logical consequence of Σ can actually be derived by using the axioms in **A** and visa versa every statement that can be derived is a logical consequence. Clearly, a necessary condition for **A** to be weakly complete is that every axiom in **A** is sound for **M**.

Intuitively, one would desire that completeness of a set of axioms **A** would imply that every sound sentence be derivable from **A**. However the definition of weak completeness does not imply such a property. Weak completeness, as will be shortly shown, only implies that sound Horn axioms are derivable from **A** but not necessarily an arbitrary non-Horn axiom. For this reason we define the following:

Definition: A set of axioms is *complete* if for every set of statements Σ and for every disjunction σ_1 *or* σ_2 *or* $\cdots$ σ_m we have

$$\Sigma \models \sigma_1 \textit{ or } \sigma_2 \textit{ or } \cdots \sigma_m \text{ iff } \Sigma \models_A \sigma_1 \textit{ or } \sigma_2 \textit{ or } \cdots \sigma_m .$$

The symbols $\models$ and $\models_A$ are defined in the same way as for single statements. When $\Sigma \models \sigma_1$ *or* σ_2 *or* $\cdots$ σ_m we say that σ_1 *or* $\cdots$ *or* σ_m is a *disjunctive consequence* of Σ.

The following lemmas present equivalent definitions for weak completeness and completeness. These concepts are taken from database theory ([Fagin 1977]). In that paper, Fagin proves a theorem similar to lemma 2, assuming that A consists of Horn axioms. The origin for this assumption is twofold: first, every universally quantified sentence in EMVD can be expressed in terms of Horn axioms (due to the *Armstrong relation* property that holds for EMVD ([Fagin 1980])). Second, it seems natural to associate weak completeness with Horn axioms, because both concepts focus on derivations of single statements. However, the restriction is not necessary as can be seen from the example of axioms (2.a) through (2.e); They are weakly complete for UGD models and, yet, include a non Horn axiom. Therefore, in our proof, the restriction of A to consist of Horn axioms is relaxed.

Lemma 2: The following conditions are equivalent.

a) A is a weakly complete set of axioms.

b) Every sound Horn axiom is derivable from A.

c) For every set of statements Σ closed under a set of sound axioms A and for every $\sigma \notin \Sigma$ there exists a dependency model M_σ that obeys all statements in Σ but does not obey σ.

Lemma 3: The following conditions are equivalent.

a) A is a complete set of axioms.

b) Every sound axiom is derivable from A.

c) For every set of statements Σ closed under a set of sound axioms A there exists a dependency model M that obeys exactly the statements in Σ.

Proof: We prove only the first lemma. The second proof is very similar and therefore omitted.

a→b: Let $A_1 : \sigma_1 \;\&\; \sigma_2 \;\&\; \cdots \;\&\; \sigma_n \Rightarrow \sigma$ be a sound Horn axiom. Let $\Sigma = \{\sigma_1, \cdots, \sigma_n\}$. Due the soundness of A_1 we have $\Sigma \models \sigma$. A is weakly complete. Therefore given Σ we can derive σ by the axioms in A. Thus A_1 is derivable from A.

b→c: Assume, by contradiction, that Σ is closed under A and that exists a statement $\sigma \notin \Sigma$ such that every model M that satisfies Σ also satisfies σ. Let $\Sigma = \{\sigma_1, \cdots, \sigma_n\}$. Consider the axiom $A_1 : \sigma_1 \;\&\; \sigma_2 \;\&\; \cdots \;\&\; \sigma_n \Rightarrow \sigma$. By our assumptions this axiom is sound for M. Clearly, $\Sigma \not\vdash_A \sigma$ because Σ is closed under A and σ is not included in Σ. Therefore, the axiom A_1 cannot be derived from A contradicting our assumption that A is weakly complete.

c→a: Assume A is a set of sound axioms that is not weakly complete in M. Thus, there exists a set Σ and a statement σ such that $\Sigma \models \sigma$ and $\Sigma \not\vdash_A \sigma$. Let $\hat{\Sigma}$ be a supper set of Σ that is closed under A and does not contain σ. Such a set always exists because $\Sigma \not\vdash_A \sigma$ (note that $\hat{\Sigma}$ is not necessarily unique, however, when A consists of Horn-axioms then $\hat{\Sigma}$ can be taken to be the unique closure of Σ under the axioms in A). By (c), Construct M_σ that satisfies $\hat{\Sigma}$ and $\neg\sigma$. M_σ also satisfies Σ and therefore σ is not logically implied by Σ, contradicting our selection of σ. □

From the above definitions it is clear that completeness implies weak completeness. The converse, however does not always hold (an example can be found in [Fagin 1977]).

Axioms (2) were found to be complete for UGD models. However, they suffer from computational disadvantages since the transitivity axiom has a disjunction on the right-hand-side of the implication. To emphasize this property we refer to such axioms as *disjunctive axioms*. The following example shows the computational disadvantages imposed by a non-Horn axiom (2.e). Moreover, it demonstrates that in some cases transitivity must be applied in order to infer a single consequence (and not only a disjunctive consequence) from a given set of independency statements.

Example: Let S=$\{I(x,z,y), I(z,wy,\gamma), I(z,wx,\gamma)\}$ be a set of three independency statements (wy stands for $w \cup y$). We show that $I(\gamma,w,z)$ is a consequence of Σ that cannot be derived without applying the transitivity axiom.

Using (2.e) on $I(x,z,y)$ we get $I(x,z,\gamma)$ *or* $I(\gamma,z,y)$. Assume $I(x,z,\gamma)$. Then using (2.d) we get $I(x,zw,\gamma)$. Adding the independency $I(z,xw,\gamma)$ (which is in S) and using (2.c) yields the independency $I(\gamma,w,xz)$ which using (2.b) yields $I(\gamma,w,z)$. On the other hand, assume $I(\gamma,z,y)$. Then using (2.d) we get $I(y,zw,\gamma)$. Adding the independency $I(z,yw,\gamma)$ (which is in S) and using (2.c) yields the independency $I(\gamma,w,yz)$ which using (1.b) yields $I(\gamma,w,z)$. Thus, $I(\gamma,w,z)$ is a single consequence of S, because it was derived by sound axioms of UGD models. It is left to show that $I(\gamma,w,z)$ cannot be derived from S without the transitivity axiom. This is done by finding the closure of S under the axioms (2.a)-(2.d) and verifying that $I(\gamma,w,z)$ is not in the closure. Indeed the closure is the set:

$$\{\, I(x,z,y),\ I(z,wy,\gamma),\ I(z,wx,\gamma),\ I(x,zw,y),\ I(x,z\gamma,y),\ I(x,zw\gamma,y),\ I(z,wyx,\gamma)\}$$

and symmetric images.

This example demonstrates two important issues. The first is that in order to apply a non-Horn axiom one needs to reason by cases, i.e., assume each term in the disjunction separately, and for each assumption reach a common conclusion. Such a process is computationally expensive because for each application of transitivity in a derivation, two new statements need to be considered independently, and each might require the use of transitivity once again. The second is: is it possible to establish a weakly complete set of Horn axioms for UGD models by replacing the transitivity axiom with a finite set of Horn axioms, say,

$$I(x,z,y)\ \&\ I(z,wy,\gamma)\ \&\ I(z,wx,\gamma) \Rightarrow I(\gamma,w,z).$$

We do not have the answer for this question; however, the example above shows the computational benefits of obtaining a complete set that consists solely of Horn axioms. Moreover, the problem of deciding whether an independency statement is implied by a given set of statements is not just of theoretical interest, rather it is a formalization of the following practical problem: Given a set of graph separation measurements (i.e., $I(x_i, z_i, y_i)$) and a single graph connectivity measurement $\neg I(u, w, v)$, it is required to determine whether these measurements are consistent. A possible paradigm for a solution would be to confirm that $I(u, w, v)$ is not implied from $\{ I(x_i, z_i, y_i) \}$. For this sake a weakly complete set of Horn axioms would have been very useful.

It is worth noting that we seek only a weakly complete set of axioms (not complete). There are two reasons for this. First, weak completeness is strong enough to assure that every single consequence is derivable, and second, a complete set of Horn axioms cannot possibly exist for UGD models.

Lemma 4: There exists no complete set of Horn axioms for UG dependency models.

Proof: Assume by contradiction that such a set, say A, exists. Examine the transitivity axiom:

$$I(x, z, y) \Rightarrow I(x, z, \gamma) \text{ or } I(\gamma, z, y)$$

Let $\Sigma=\{I(x, z, y)\}$. Since transitivity is a sound axiom for UGD models, clearly $I(x, z, \gamma)$ *or* $I(\gamma, z, y)$ is a consequence of Σ. Examine a derivation of this disjunction (there exists one because A is complete). The assumption that all axioms in A are Horn-axioms implies that to derive this disjunction one of the individual statements would have been derived first, suggesting that one of the individual statements is logically implied by Σ. This is clearly not so, thus A could not have been complete. □

One of the results of this thesis is showing that, while the task of finding a weakly complete finite set of Horn axioms for the UGD model is worthwhile, a similar task for DAGD models cannot be fruitful since such a set does not exist.

A recent result, reported in [Verma 1987a], states that the following axioms are complete for DAG-isomorph dependency models.

(weak transitivity)

$$I(x, z, y) \,\&\, I(x, z \cup \gamma, y) \Rightarrow I(x, z, \gamma) \text{ or } I(\gamma, z, y) \tag{3.a}$$

(chordality)

$$I(\alpha, \{\gamma, \delta\}, \beta) \,\&\, I(\gamma, \{\alpha, \beta\}, \delta) \Rightarrow I(\alpha, \gamma, \beta) \text{ or } I(\alpha, \delta, \beta) \tag{3.b}$$

$$(3.c) \quad I(X, Z, Y) \;\; iff \;\; \forall_{x \in X,\, y \in Y}$$

$$\left\{ \exists_{S \subset U - xy} I(x, S, y) \; or \; I(y, S, x) \right\}$$

$$and$$

$$\left\{ \forall_{c \in Z} \exists_{S \subset U - xy} I(x, Sc, y) \; or \; I(y, Sc, x) \right\}$$

$$and$$

$$\left\{ \begin{array}{c} \forall_{a, b \in U} I(x, Z, b) \; or \; I(a, Z, y) \; or \; \left[\forall_{S \subset U - ab} \neg I(a, S, b) \text{ and } \neg I(b, S, a) \right] \\ or \\ \left[\forall_{c \in Z} \exists_{S \in U - ab} I(a, Sc, b) \; or \; I(b, Sc, a) \right] \end{array} \right\}$$

$$and$$

$$\left\{ \begin{array}{c} \forall_{a \in U} I(a, Z, y) \; or \; \left[\exists_{S \subset U - ax} I(x, S, a) \; or \; I(a, S, x) \right] \\ or \\ \left[\exists_{w \in U} \left[\exists_{S \in U - wx} I(x, S, w) \; or \; I(w, S, x) \right] \text{ and } \left[\forall_{S \subset U - wx} \neg I(x, Sa, w) \text{ and } \neg I(w, Sa, x) \right] \right] \end{array} \right\}$$

$$and$$

$$\left\{ \begin{array}{c} \forall_{b \in U} I(x, Z, b) \; or \; \left[\exists_{S \subset U - by} I(b, S, y) \; or \; I(y, S, b) \right] \\ or \\ \left[\exists_{w \in U} \left[\exists_{S \in U - wy} I(w, S, y) \; or \; I(y, S, w) \right] \text{ and } \left[\forall_{S \subset U - wy} \neg I(w, Sb, y) \text{ and } \neg I(y, Sb, w) \right] \right] \end{array} \right\}$$

These axioms express the d-separation criteria in terms of independency statements, while defining precisely the restrictions that $I(x, z, y)$ should obey in order to be perfectly represented with a DAG. Axioms (3) cannot be used as a derivational tool because each application of axiom (3.c) requires exponential (in the number of variables) amount of work. The next two sections, though, show that this is probably the best that we can hope and that these complex axioms cannot be replaced with an equivalent set of Horn axioms or even disjunctive axioms.

The following axioms are a partial (i.e., not complete) list of sound axioms for DAGD models and are taken from [Pearl 1986a]:

(symmetry)

$$I(x, z, y) \Longleftrightarrow I(y, z, x) \tag{4.a}$$

(decomposition-composition)

$$I(x, z, y \cup w) \Longleftrightarrow I(x, z, y) \,\&\, I(x, z, w) \tag{4.b}$$

(intersection)

$$I(x, z \cup w, y) \,\&\, I(x, z \cup y, w) \Rightarrow I(x, z, y \cup w) \tag{4.c}$$

(weak union)

$$I(x, z, y) \Rightarrow I(x, z \cup w, y) \tag{4.d}$$

(weak transitivity)

$$I(x, z, y) \,\&\, I(x, z \cup \gamma, y) \Rightarrow I(x, z, \gamma) \text{ or } I(\gamma, z, y) \tag{4.e}$$

(chordality)

$$I(\alpha, \{\gamma, \delta\}, \beta) \,\&\, I(\gamma, \{\alpha, \beta\}, \delta) \Rightarrow I(\alpha, \gamma, \beta) \text{ or } I(\alpha, \delta, \beta) \tag{4.f}$$

The axioms in (4) are all derivable from axioms (3.a) through (3.c) which are complete, however, they are far easier to prove directly from the properties of d-separation. These axioms are listed to facilitate the discussion in subsequent sections.

4. NON AXIOMATIZABILITY OF DAGS

In this section we prove the incompleteness theorem. The theorem states that there exists no finite weakly complete (nor complete) set of Horn axioms for DAGD models with the d-separation criteria. We use two lemmas (5 & 6) in the proof. In lemma (5) we present a Horn axiom, called $R_0(n)$, with n antecedents, denoted S, and prove that $R_0(n)$ is sound for DAGD models. In lemma (6) we list all single consequences of S (for $n \geq 7$) and prove that neither of them is a single consequence of any proper subset of S. The proof of the incompleteness theorem then follows using the following argument:

Define the *arity* of an axiom to be the number of antecedents of that axiom. Using this definition, lemma 6 states that for $n \geq 7$ the axiom $R_0(n)$ is irreducible to a chain of smaller arity Horn axioms. We proceed by contradiction. Assume A is a complete finite set of Horn axioms. Let k be the largest arity of all the axioms in A. Pick $n = max(k+1, 7)$. Consider the axiom $R_0(n)$. By lemma 6, this axiom is irreducible to a chain of smaller arity Horn axioms. All axioms in A have smaller arity then the arity of $R_0(n)$. Therefore, the consequence of $R_0(n)$ can not be derived from its antecedents by using only the axioms in A. This contradicts our assumption that A is weakly complete.

We now prove the two lemmas.

Lemma 5: The axiom $R_0(n)$:

$$I(A_2,A_1,A_3) \;\&\; I(A_3,A_2,A_4) \;\&\; \cdots \; I(A_n,A_{n-1},A_{n+1}) \;\&\; I(A_{n+1},A_n,A_1) \Rightarrow I(A_{n+1},\varnothing,A_n \cup A_1)$$

holds in DAGD models.

Proof (By contradiction): Assume $R_0(n)$ does not hold. Consider a DAG that obeys all antecedents of $R_0(n)$ and does not obey $I(A_{n+1}, \varnothing, A_n \cup A_1)$. In this DAG, there exists a path between an element α_{n+1} of A_{n+1} and an element β of $A_n \cup A_1$, that is not d-separated by $\emptyset$. Clearly, this path does not contain a head-to-head node. Let $P = (\alpha_{n+1}, \beta)$ be the shortest such path. Two cases need to be examined; β belongs to A_1 and β belongs to A_n.

Case 1: $P = (\alpha_{n+1}, \alpha_1)$ where α_1 belongs to A_1. The statement $I(A_{n+1}, A_n, A_1)$ implies that A_n d-separates the path P. P does not contain a head-to-head node and therefore, in order to block two successive arrows, an element α_n of A_n must reside on P. Consider the path (α_{n+1}, α_n). It does not contain a head-to-head node because P does not contain such node. Hence, (α_{n+1}, α_n) is a path from A_{n+1} to $A_n \cup A_1$ that is not d-separated by $\emptyset$. This path contradicts our definition of P, because it is shorter then P.

Case 2: $P=(\alpha_{n+1}, \alpha_n)$ where α_n belongs to A_n. the statement $I(A_{n+1}, A_{n-1}, A_n)$ implies that an element α_{n-1} of A_{n-1} is on P. Consider the path (α_{n-1}, α_n). This path has no head-to-head nodes, therefore the statement $I(A_{n-1}, A_{n-2}, A_n)$ implies that an element α_{n-2} of A_{n-2} must reside on (α_{n-1}, α_n). Similarly, an element of each of the sets $A_{n-3}, A_{n-4}, \ldots, A_1$ must reside on P. Consider the path (α_{n+1}, α_1) where α_1 is an element of A_1 that is on P. This path contradicts our definition of P because it is shorter then P. □

Lemma 6: Let S be all the antecedents of $R_0(n)$. Then the only non-trivial single consequences of S for $n \geq 7$ are: $I(A_{n+1}, \emptyset, A_n \cup A_1)$, $I(A_{n+1}, A_1, A_n)$, $I(A_{n+1}, \emptyset, A_n)$, $I(A_{n+1}, \emptyset, A_1)$ none of which is a consequence of any proper subset of S.

Proof: Let VALID be the set $\{ I(A_{n+1}, \emptyset, A_n \cup A_1), I(A_{n+1}, A_1, A_n), I(A_{n+1}, \emptyset, A_n), I(A_{n+1}, \emptyset, A_1)\}$. From lemma 5 we know that $I(A_{n+1}, \emptyset, A_n \cup A_1)$ is a consequence of S while the other three statements in VALID can immediately be derived from it using decomposition (4.b) and the weak union (4.d) axioms.

Assume $I(x, z, y)$ is an arbitrary non trivial independency statement not in **VALID**. We will show that $I(x, z, y)$ is not a consequence of S by constructing a DAG that obeys S and does not obey $I(x, z, y)$. Without loss of generality assume that the A_i's are singletons, $A_i=\{a_i\}$ $i=1..n+1$ and that these are all the nodes of the DAG (It suffices to contradict the axiom for any assignment of A_is). Also w.l.o.g, x and y can be considered as singletons due to the decomposition axiom (4.b):

$$I(x_1, z, y_1 \cup y_2) \Rightarrow I(x_1, z, y_1) \,\&\, I(x_1, z, y_2).$$

Namely, if $I(x_1, z, y_1)$ is not a possible consequence then neither is $I(x, z, y)$ where x, y are any sets containing x_1 and y_1 respectively. For any assignment of x, y and z, our task is to construct a DAG that satisfies all the n antecedents of $R_0(n)$ but violates $I(x, z, y)$.

Assume $x=A_j$, $y=A_k$, $j<k$ and examine the statement $I(A_j, Z, A_k)$ for all Z and for all possible values of j and k. We will say that j and k are *consecutive* if *abs[j-k mod(n+1)]=1*. All subscript expressions here will be taken modulo $n+1$. For example: $k=j+1$ when $j<n+1$ and $k=1$ when $j=n+1$. Also, to clarify the figures, we label the nodes i, j,... instead of a_i, a_j,....

Case 1: j and k are not consecutive.

Examine the following DAG :

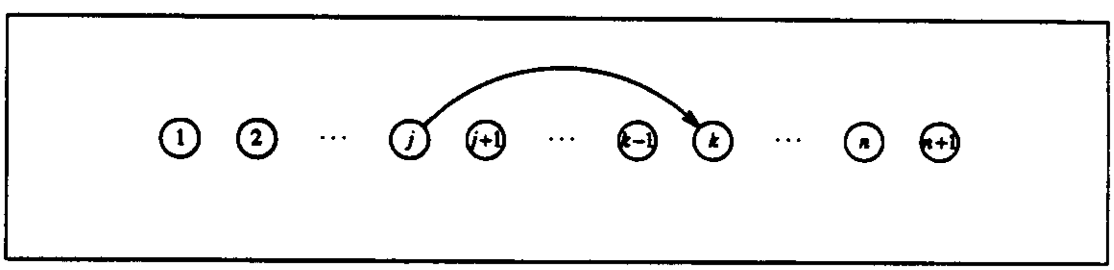

Figure 2

$I(A_i, A_{i-1}, A_{i+1})$ holds for all i because i and $i+1$ are always disconnected.

However, the proposed conclusion $I(A_j, Z, A_k)$ is false for any set Z (including $\varnothing$), because j and k are connected with a direct link.

Case 2: j and k are consecutive.

Subcase 2.1: Z contains a variable i which is not consecutive to either j or k.

Examine the following DAG:

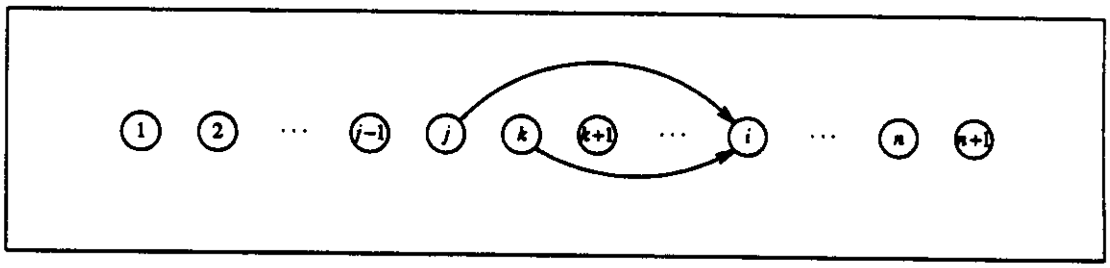

Figure 3

$I(A_j, Z, A_k)$ is clearly false. Yet, all independencies in S hold because i is not consecutive to either j or k.

Subcase 2.2: Z is a subset of $A_{j-1} \cup A_{k+1}$.

Subcase 2.2.1: $Z = A_{j-1} \cup A_{k+1}$

Construct The DAG of Figure 4 where i is an arbitrary node other then $j-2, \ldots, k+1, k+2$. Such an i always exists for $n \geq 7$. Once again, S holds, but $I(A_j, A_{j-1} \cup A_{k+1}, A_k)$ is false, because $A_{j-1} \cup A_{k+1}$ activate a path $(j, k+1, i, j-1, k)$ between j and k.

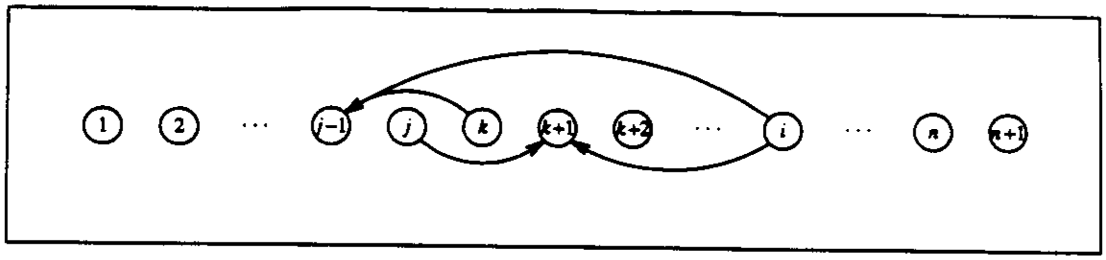

Figure 4

Subcase 2.2.2: $Z = A_{k+1}$.

The DAGs of Figures 5 and 6, realize S & $\neg I(A_j, A_{k+1}, A_k)$ for the cases $2 < j < n$, and $j = n+1$ or 1 or 2, respectively.

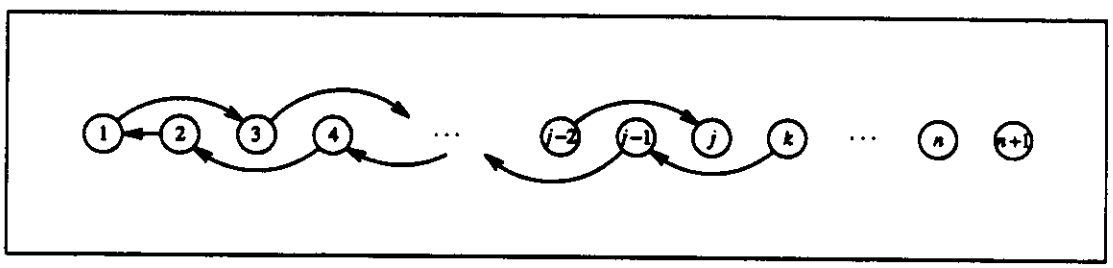

Figure 5

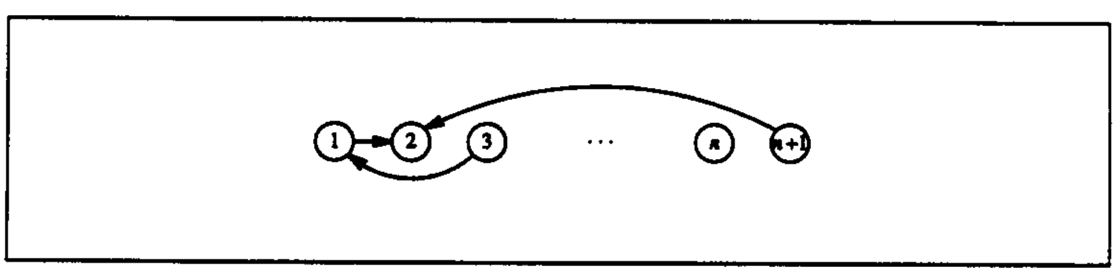

Figure 6

In case $j = n$, $I(A_j, A_{k+1}, A_k)$ reduces to $I(A_n, A_1, A_{n+1})$ which is a member of **VALID**.

Subcase 2.2.3: $Z = A_{j-1}$.

For $j = 1$ the DAG of Figure 6 realizes S & $\neg I(A_1, A_{n+1}, A_2)$, while for all other values of j, the statement $I(A_j, A_{j-1}, A_k)$ is a trivial consequence because it is a member of S.

Subcase 2.2.4: $Z = \emptyset$

For $j < n$ the DAG of figure 5 realizes S & $\neg I(A_j, \emptyset, A_k)$, while $j = n$ and $j = n+1$ yield $I(A_n, \emptyset, A_{n+1})$ and $I(A_{n+1}, \emptyset, A_1)$, which are in **VALID**.

So far, we have shown that all single non trivial consequences of S are listed in **VALID**. To complete the proof, we have to verify that every statement in **VALID** cannot be inferred from any proper subset of S.

Consider the statement $I(A_n, A_1, A_{n+1})$ that belongs to **VALID**. Let S' be a proper subset of S. The following DAG satisfies any S' not containing $I(A_{n+1}, A_n, A_1)$ but does not satisfy the consequence $I(A_n, A_1, A_{n+1})$.

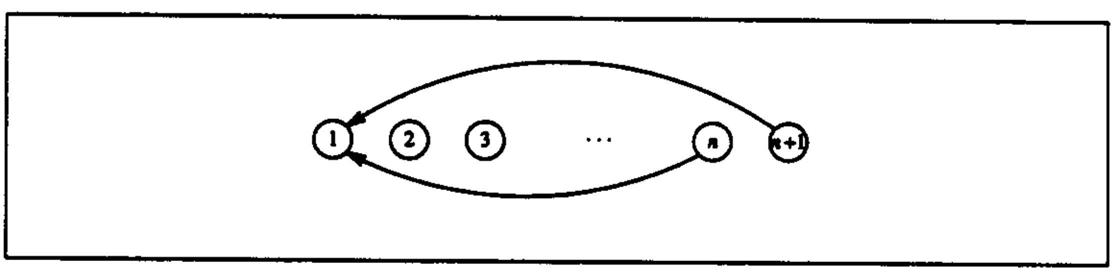

Figure 7

The other possibility is that $I(A_{n+1}, A_n, A_1)$ is included in S' but one or more of the first n-1 independencies is not. Define m to be the maximal i such that $I(A_{i+1}, A_i, A_{i+2})$ is not in S'. Again, the following DAG satisfies S' but not the consequence $I(A_n, A_1, A_{n+1})$ (When $m = n-1$, only one link between n and $n+1$ is needed).

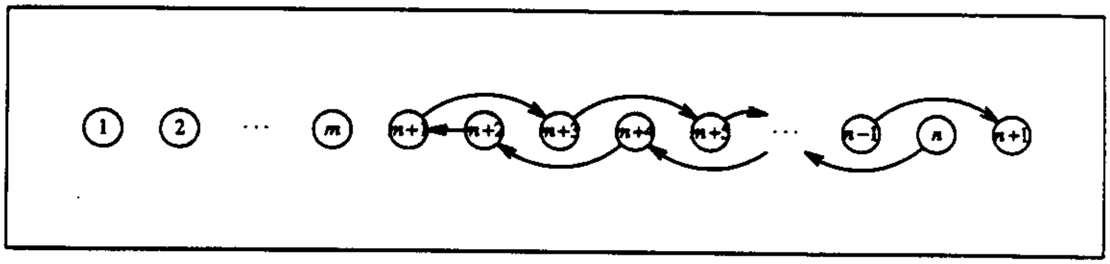

Figure 8

Following similar arguments, the other three statements in **VALID**, can also be shown to require all n antecedents of $R_0(n)$. □

We have completed the proof of the following theorem:

Theorem 7: There exists no finite weakly complete (nor complete) set of Horn-axioms for DAGD models.

5. EXTENSIONS OF THE INCOMPLETENESS THEOREM

The non existence of a finite complete set of Horn axioms for DAGD models, still leaves us with the question of whether one could establish a finite complete set of *disjunctive* axioms for DAGD models (like that of UGD models). Such a set, though, would most likely be computationally intractable and would be useful only as a theoretical tool for studying the properties of the $I(x, z, y)$ relation in the DAGD model. We conjecture that even disjunctive axioms would not render DAGD models axiomatizable, i.e. a complete finite set of disjunctive axioms does not exist. We emphasize that this conjecture does not exclude the possibility of a finite weakly complete set of disjunctive axioms for DAGD models. The concern of this section is only in the existence of a complete set of disjunctive axioms while the existence of a weakly complete set remains an open problem.

Conjecture: There exists no bounded complete set of disjunctive axioms for DAGD models with the d-separation criteria.

Although, we have not been able to prove this conjecture in general, it can be shown to hold for a large subset of disjunctive axioms.

Before proceeding, we need the following classification of axioms. An axiom

$$I(x_{1,1}, x_{1,2}, x_{1,3})\ \&\ \cdots\ I(x_{n,1}, x_{n,2}, x_{n,3}) \Rightarrow I(y_{1,1}, y_{1,2}, y_{1,3})\ or\ \cdots\ I(y_{m,1}, y_{m,2}, y_{m,3})$$

is a ***functional-restricted*** axiom if every set $y_{i,j}$ is a result of applying boolean functions on the sets $x_{i,j}$. Namely, each $y_{i,j}$ is the result of applying the set-functions: union, intersection and negation on the $x_{i,j}$'s. For example, the weak transitivity axiom:

$$I(x, z, y)\ \&\ I(x, z \cup \gamma, y) \Rightarrow I(x, z, \gamma)\ or\ I(\gamma, z, y)$$

is a functional-restricted axiom because all the arguments on the right-hand-side of the implication are functionally dependent on the arguments on the left-hand-side. Specifically, γ that can be written as $(z \cup \gamma) \cap \neg z$ where both z and $z \cup \gamma$ appear on the left-hand-side of the implication. On the other hand, the transitivity axiom (which does not hold for DAGD models):

$$I(x,\ z,\ y) \Rightarrow I(x,\ z,\ \gamma)\ or\ I(\gamma,\ z,\ y)$$

is not a functional-restricted axiom because γ is not functionally dependent on x, y and z.

Theorem 8: There is no bounded complete set of functional-restricted axioms for DAGD models with the d-separation criteria.

Remark: Theorem 8 does not exclude the possibility of a weakly complete set of functional-restricted axioms for DAGD models, the existence of which remains an open problem.

proof: This proof uses similar techniques to the ones used in the proof of theorem 7. Consider the axiom $R_0(n)$ of lemma 5. It is sufficient to prove that this axiom can not be reduced to a chain of lower arity functional-restricted axioms. The reason that the proof of lemma 6 can not be immediately applied is that there, we only showed that all single consequences need the full set of antecedents S in order to be concluded. However now, we need to show that all disjunctive consequences has this property as well, else a disjunctive consequence might turn $R_0(n)$ reducible to disjunctive axioms. Thus, for each disjunction of independency statements π that is not a disjunctive consequence of S, one needs to find a DAG that obeys S but does not obey the disjunction π. Such a task is infeasible because the length of the disjunction in π is arbitrary and an infinite number of constructions are needed.

To overcome this problem we use, for each term T_i in π, a DAG called D_i that obeys S but does not obey T_i. Then we define an operation that collapses the sequence of D_i's to a single DAG denoted $\oplus D_i$ that obeys S but does not obey π. This construction, as will be shown latter, is made possible due to the restriction on the axioms to consist solely of functional-restricted axioms. $\oplus D_i$ is the required DAG showing that π is not a disjunctive consequence of S.

Let A_k be the set $\{a_{k,1}, a_{k,2}, \cdots a_{k,m}\}$ where $a_{i,j}$ are single elements and let

$$\pi = I(x_1,z_1,y_1) \text{ or } I(x_2,z_2,y_2) \text{ or } \cdots I(x_m,z_m,y_m)$$

be an arbitrary disjunction of statements using the same attributes of S. Assume that all terms in π are not members of **VALID**, otherwise trivialy, π is a disjunctive consequence of S.

The restriction on the axioms to be functional-restricted constrains the disjunctions π that need to be examined. Namely, the sets x_i's, y_i's and z_i's are all functionally dependent on the A_k's. However, the A_k's are a partition on the nodes of $\oplus D_i$ and therefore negation and intersection can be reexpressed in terms of union only. Thus w.l.o.g, we can assume that each of the x_i's, y_i's and z_i's is a union of some A_k's. Moreover, due the decomposition axiom, we can further assume that x_i and y_i are each equal exactly to some A_k. This is shown in the following argument: if π is not obeyed by some DAG D, then none of its terms are obeyed. By the decomposition axiom, each term can be augmented by a set w_i and the resulting disjunction,

$$\pi' = I(x_1,z_1,y_1 \cup w_1) \text{ or } I(x_2,z_2,y_2 \cup w_2) \text{ or } \cdots I(x_m,z_m,y_m \cup w_m)$$

is not obeyed by D. Thus a DAG not satisfying π does not satisfy π' as well.

Consider the following construction of $\oplus D_i$:

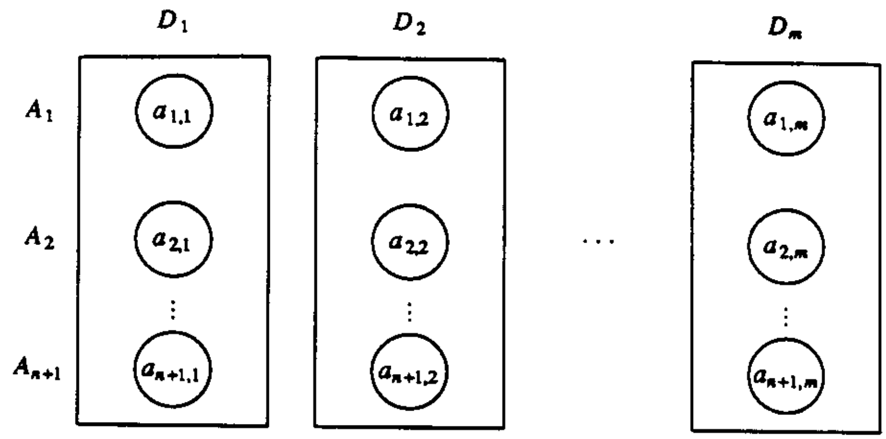

Figure 9: Construction of $\oplus D_i$.

$\oplus D_i$ is a collection of m disconnected components D_i, one for each term in π. The nodes of each D_i, denoted N(D_i), are labeled $\{ a_{1,i}, a_{2,i}, \cdots a_{n+1,i} \}$. Thus, each component D_i contains exactly one variable from every A_k (namely, $a_{k,i}$). In the following discussion the term, $I(x, z, y)$ *holds in* D_i, means that

$I(x \cap N(D_i), z \cap N(D_i), y \cap N(D_i))$ holds in D_i.

D_i is constructed as in lemma 5, in such a way that S holds in D_i and the i-th term of π (namely, $I(x_i, z_i, y_i)$) does not hold. For example, if $I(x_i, z_i, y_i) = I(A_1, A_5, A_3)$ then we use the DAG of Figure 2 so that all members of S hold in D_i and $I(a_{1,i}, a_{5,i}, a_{3,i})$ does not hold. Hence, we have used those elements which are in D_i to construct a path between A_1 and A_3 that is d-separated by $a_{5,i}$. This path is also d-separated by A_5 because all other elements of A_5 reside in components which are disconnected from D_i (This claim is made more rigorous in lemma 9).

This construction is made possible because the x_i's, y_i's and z_i's are each a union of some A_k's and therefore **have elements in each component** D_i. These elements are needed to establish a path, entirely within D_i that is not d-separated by z_i.

For example, assume that $I(x_1, z_1, y_1) = I(A_{n+1}, \{a_{2,2}\}, A_1)$ (Note that $a_{2,2}$ is not functionally dependent on the A_k's). For this term our construction is not adequate. The elements of D_1 are not sufficient to realize S and $\neg I(a_{n+1,1}, a_{2,2}, a_{1,1})$, because $I(a_{n+1,1}, \emptyset, a_{1,1})$ (and $I(A_{n+1}, \emptyset, A_1)$) is a consequence of S. Therefore, a link must be drawn between D_1 and D_2. This destroys the disconnectness of the components D_i and therefore, as is shortly shown, no longer can we prove that $\oplus D_i$ obeys S. It should be emphasized, though, that for each disjunction π, even when such terms are present, it is easy to construct the required DAG. This is the reason for our belief that DAGD models are indeed non axiomatizable with disjunctive axioms. However, in general, we could not establish the proof without the restriction on the axioms that excluded the need to consider such terms.

We now prove that $\oplus D_i$ satisfies S and not π. For this purpose, we present the following lemma (the proof is given latter).

Lemma 9: Let D be a DAG that consists of m disconnected components D_i and let V_i be the nodes of D_i. Then, the following two statements are equivalent.

(9.a) $I(x, z, y)$ holds in D

(9.b) For every i, the projection of $I(x, z, y)$ on D_i holds, namely, the statement $I(x \cap V_i, z \cap V_i, y \cap V_i)$ holds in D_i.

We employ this lemma in two ways. First, by our construction, every member of S satisfies (9.b) therefore every member of S holds in $\oplus D_i$. Second, each term T_i of π has one component in which T_i does not hold and therefore T_i does not hold in $\oplus D_i$. Hence, π does not hold in $\oplus D_i$.

To complete the proof of theorem 8 we argue that if π has a term T_{i_0} that is a member of **VALID** then it cannot be inferred from any proper subset S' of S. This is evidently true because for each subset S', a DAG was constructed (Figures 6 & 7) that obeys S' and not T_{i_0}. The use of these DAGs in the same fashion described earlier, shows that π is not a disjunctive consequence of any proper subset of S. Thus, the axiom $R_0(n)$ is irreducible to a chain of functional-restricted axioms of lower arity then n. □

Proof of lemma 9: Define $x_i \equiv x \cap V_i$, $y_i \equiv y \cap V_i$ and $z_i \equiv z \cap V_i$. In words, x_i means the *projection* of x on the nodes of the ith component. Note that since V_i are partitioning the nodes of D, we have $x = \bigcup_i x_i$, $y = \bigcup_i y_i$ and $z = \bigcup_i z_i$.

(a)⇒(b): Assume $I(x, z, y)$. We prove that $I(x_i, z_i, y_i)$ holds for all i. Due to the decomposition axiom (4.b), $I(x, z, y)$ implies that $I(x_i, z, y_i)$ also holds. That is, z d-separates all paths between x_i and y_i. It remains to show that also z_i d-separates these two sets. Two cases need to be examined; first, if a path is d-separated because the existence of an element of z on it. Then, this element must also be a member of z_i because D_i is a disconnected component. Second, if a path is d-separated because none of the elements of z reside on it (i.e a head to head node) then the removal of some elements of z leaves this path d-separated.

(a)⇐(b): Assume $I(x_i, z_i, y_i)$ holds for all i. Let u and v be two sets each contained in a different disconnected component of D. Then clearly, for each set w the statement $I(u, w, v)$ holds in D. In particular, assign $u = x_i$, $w = z_i$ and $v = z_j$ $j \neq i$. The statement $I(x_i, z_i, z_j)$ holds for every j, $j \neq i$. Using the composition axiom (4.b) we obtain $I(x_i, z_i, (\bigcup_{j \neq i} z_j) \cup y_i)$. Applying the weak union property we obtain $I(x_i, \bigcup_i z_i, y_i)$ which reduces to $I(x_i, z, y_i)$. We now use the assignment $u = x_i$, $w = z$ and $v = y_j$ $j \neq i$ to obtain

$I(x_i, z, y_j)$ for every j, $j \neq i$. Using composition we obtain $I(x_i, z, y)$. This statement holds for every i, therefore applying the composition axiom again, we obtain that $I(x, z, y)$ holds. □

Note that lemma 9 states that, when a reasoning system based on DAGs is composed of components that are disconnected, then the reasoning can be done separately in each component.

The next lemma shows that the construction of theorem 8 is made possible only because we restricted the allowed axioms and thereby restricted the domain of disjunctions of independency statements that need to be examined. In other words, there is no way to construct an operator $\oplus$ that produces from a sequence of arbitrary DAGs D_i, a DAG for which an independency statement holds iff it holds for each D_i. This lemma is a restricted version of a theorem found in ([Fagin 1980]). The original theorem is stated in a more general terminology that is useful for both relational database theory and Dependency models theory. We supply only the proof of the part that we use, because it is closely related to the construction of of theorem 8.

Lemma 10: Let **M** be a set of dependency models. The following properties of **M** are equivalent.

(10.a) There is an operation $\oplus$ that maps families of models into models, such that if σ is an independency statement, and if $\{ M_i : i \in I \}$ are models of **M**, then σ holds for $\oplus < M_i : i \in I >$ iff σ holds for each M_i.

(10.b) Whenever Σ and $\{ \sigma_i : i \in I \}$ are sets of independency statements, then σ_1 *or* σ_2 *or* $\cdots$ σ_n is a consequence of Σ iff there exists an i such that σ_i is a consequence of Σ.

Proof:

(a)⇒(b): By contradiction assume σ_1 *or* σ_2 *or* $\cdots \sigma_m$ is a consequence of Σ and that each σ_i is not a consequence of Σ. Then for each σ_i there exist a model M_i that obeys Σ but does not obey σ_i. Consider the model $\oplus M_i$. This model obeys Σ but does not obey any σ_i, contradicting our assumption that σ_1 *or* σ_2 *or* $\cdots \sigma_i$ is a consequence of Σ. □

The application of this lemma for DAGD models is straight forward. Consider the weak transitivity axiom (4.e) that holds in DAGs. This axiom does not satisfy the conditions in (10.b). Lemma 10 assures us that an operation $\oplus$ that meets the requirements in (10.a), does not exist for DAGD models.

A dependency model obeying the conditions of lemma 10 is called an *Armstrong relation* ([Fagin 1980]). For Armstrong relations the concepts of completeness and weak completeness are equivalent. This is due to (10.b) which suggests that σ_1 *or* σ_2 *or* $\cdots \sigma_m$ is logically implied from Σ only if some σ_i is implied from Σ. Moreover, if a complete (weakly complete) set of axioms exists for an Armstrong relation then an equivalent complete (weakly complete) set of Horn axioms must exist as well.

This observation can be exploited to extend the result reported in [Parker and Parsaye 1980]. They have proven that there is no finite weakly complete (nor complete) set of Horn axioms for EMVD models. However, EMVD models are Armstrong relations, because there exists an operation $\oplus$ that satisfies the requirements of lemma 10 ([Beeri & Fagin 1977],[Fagin 1980]). Therefore, we can also conclude that there is no finite weakly complete (nor complete) set of disjunctive axioms for the EMVD model. This result was never explicitly stated neither in [Parker & Parsaye 1980] nor in [Sagiv & Walecka 1982] perhaps because it was presummed to be self-evident.

6. MULTI-GRAPHS DEPENDENCY MODELS

The inability of DAG and UG dependency models to fully represent an arbitrary semi-graphoid has lend to the following generalization: Instead of one graph over a set of nodes we consider a collection of graphs.

Definition : [Paz 1987] A Multi Undirected Graph dependency model (MUG) N is a collection of m undirected graphs over a set of nodes U with the following modified vertex-separation criteria:

$I(x, z, y)$ holds for N iff there exists a graph G in N that satisfies $I(x, z, y)$

G satisfies $I(x, z, y)$ iff (1) Z is a cutset of x and y in G

(2) The vertices of G are contained in $x \cup y \cup z$

As the next example shows, the second condition is essential because a single graph in a MUG might contain a partial set of U as its nodes.

Example:

Figure 10: An MUG displaying $I(1, 3, 2)$ and $I(1, 4, 2)$

In this example $I(1,3,2)$ and $I(1,4,2)$ are the only non-trivial statements that hold while $I(1,34,2)$ does not hold (due the second condition of the separation criteria), thus, the strong union axiom (2.c) does not hold in MUGs.

A similar definition of MDAGs is obtained by modifying the d-separation criteria as follows:

Definition: Let N be an MDAG.

$I(x, z, y)$ holds for N iff there exists a DAG D in N that satisfies $I(x, z, y)$

D satisfies $I(x, z, y)$ iff (1) Z d-separates x from y in D

(2) The vertices of D are contained in $x \cup y \cup z$

The main interest in these classes of models is their ability to perfectly represent an arbitrary semi-graphoid. This is non trivial, because semi-graphoids involve set-to-set relationships while graph separation is a property that is made up from separation of singleton nodes, i.e., the composition axiom (4.b) holds.

Definition: A dependency model M is said to be a *multi-graph-isomorph* if there exists a multi graph $MG = \{(U_i, E_i)\}$ which is a perfect map of M, i.e., for every three disjoint subsets x, y and z of U, we have:

$$I(x, z, y)_M \iff I(x, z, y)_{MG}$$

Theorem 11: A necessary and sufficient condition for a dependency model M to be multi-graph-isomorph is that $I(x, z, y)_M$ satisfies the three (independent) axioms (the subscript M dropped for clarity):

(symmetry)

$$I(x, z, y) \Rightarrow I(y, z, x) \quad (5.a)$$

(decomposition)

$$I(x,\ z,\ y \cup w) \Rightarrow I(x,\ z,\ y)\ \&\ I(x,\ z,\ w) \tag{5.b}$$

(weak union)

$$I(x,\ z,\ y \cup w) \Rightarrow I(x,\ z \cup w,\ y) \tag{5.c}$$

Proof: We first show that the axioms in (5) are obeyed by MUGD models. This is due to the following three properties of these axioms: first, they hold for UGD models. Second, they are all *unary* (i.e., have only one antecedent) and third, they do not introduce any new elements in the right-hand-side of the implication. The first property is necessary because a MUG might consist of a single graph. The second property guarantees that whenever the left hand side of an axiom is obeyed by some MUG N then it is obeyed by a single graph of N. (contrary to the Contraction axiom (1.d) where each of the antecedents might be obeyed by a different graph of N, in which case, the consequence need not be represented). Due to the third property, any graph that obeys the antecedent of a unary axiom must also obey the consequence (unlike the strong union axiom (2.d) where the introduction of new elements renders this axiom unsound for MUGD models as shown in Figure 10). We note that the arguments for the soundness of axioms (5) for MUGD models are applicable, without change, also for MDAGD models.

The second part of the proof is the construction of a MUG that is a perfect map of M. This is done in an adaptive manner; for each independency statement σ of M we construct an undirected graph G_σ that satisfies only σ and statements derivable from σ. In other words, if we define $cl(\sigma)$ to be the closure of σ under the axioms in (5) then our requirement is that G_σ is a perfect map of $cl(\sigma)$. To complete the proof we will prove that the constructed collection of G_σ is the desired MUG representation of M.

Let $\sigma = I(x,\ z,\ y)$. G_σ is constructed by removing the links (α, β) where $\alpha \in x$ and $\beta \in y$, from the complete graph (over the variables in $x \cup y \cup z$). What remains are three cliques representing the sets x, y and z and the links between the cliques of x, z and y, z. This construction is demonstrated in figure 11.

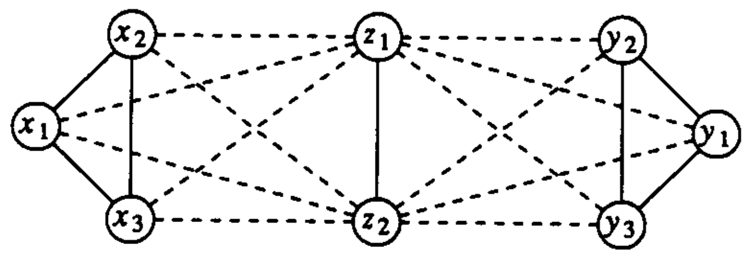

Figure 11: The construction of $G_{I(\{x_1\, x_2\, x_3\},\ \{z_1\, z_2\},\ \{y_1\, y_2\, y_3\})}$

Our claim is that G_σ is a perfect map of $cl(\sigma)$. First we show that if $\sigma' \in cl(\sigma)$ then σ' holds for G_σ. By definition of $cl(\sigma)$, we know that σ' is derivable from σ. Let $\sigma = \sigma_0,\ \sigma_1,\ \sigma_2,\ \cdots\ \sigma_n = \sigma'$ be a sequence of statements generated in a derivation of σ', where each σ_i in the sequence is derived for the previous statements by the axioms in (5). Since σ_0 is obeyed by G_σ and because each of the statements were derived using axioms that hold in every UGD model then by induction, every σ_i holds in G_σ. In particular, σ' holds G_σ.

The converse, namely, that σ' holds for G_σ implies $\sigma' \in cl(\sigma)$ involves a characterization of all statements that hold in G_σ and show that all of them are derivable from σ.

Let $I(u,\ v,\ w)$ be an arbitrary statement over the variables in G_σ. We assume that u and w are not empty, because, by definition, $I(u,\ v,\ \emptyset)$ holds in every dependency model and therefore derivable from σ. We show that if $I(u,\ v,\ w)$ holds then it must be of the form $I(x'',\ z \cup x' \cup y',\ y'')$, where $x'\ x''$, $y'\ y''$ are subsets of x and y respectively.

If u contains an element of x then w must consist only from elements of y, because otherwise there would be a path from u to w. Since w consists of elements from y, u must be a subset of x. We are left with a statement of the form $I(x'', x' \cup y' \cup z', y'')$. This statement holds iff all paths from x'' to y'' are blocked and this happens only when $z' = z$. The resulting statement $I(x'', z \cup x' \cup y', y'')$ is clearly derivable from σ. If u or w contains an element of z then $I(u, v, w)$ does not hold, because every element in z is connected to all the nodes in the graph. All other cases, namely if u contains an element of y, w contains an element of x or w contains an element of y can be proven identically, because of the symmetric roles of x, y and u, w respectively.

It remains to argue that the collection of these graphs $G_\Sigma = \{G_\sigma \mid \sigma \in M\}$ is a perfect map of M. By our construction, every statement σ' that holds in G_Σ holds also in some graph $G_\sigma \in G_\Sigma$. Since G_σ is a perfect map of $cl(\sigma)$ we know that σ' is derivable from σ and therefore holds in M. Thus, G_Σ is an I-map of M. The other direction holds as well. Every statement σ that holds in M has a graph in G_Σ where it holds and therefore σ holds in G_Σ. Thus G_Σ is a perfect map of M. □

Identical result can be proven for MDAGD models. Dependency models that have a perfect MDAG representation are called ***multi-DAG-isomorph*** and are also fully characterized with the axioms in (5). The only change in the proof of theorem 11 is that instead of an undirected graph G_σ we construct a DAG, called D_σ.

The construction of D_σ is very similar and the only change required is to present directionality on G_σ. The directions are added in the following way; For every $\alpha \in x$, $\beta \in y$ and $\gamma \in z$, the links (α, γ) become $\alpha \rightarrow \gamma$ and The links (γ, β) become $\gamma \rightarrow \beta$. All other links, are within the cliques x, y and z, and are assigned arbitrary directions that leave the corresponding cliques acyclic. This construction is demonstrated in the next figure.

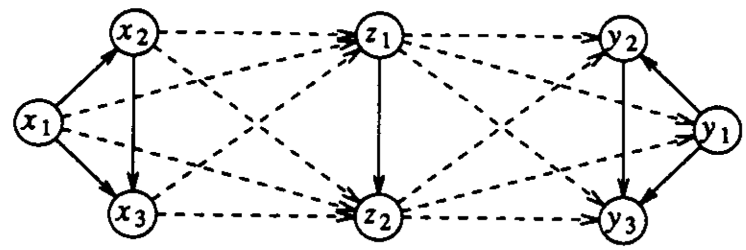

Figure 12: The construction of $D_{I(\{x_1\, x_2\, x_3\}\,,\ \{z_1\, z_2\}\,,\ \{y_1\, y_2\, y_3\})}$

The proof of theorem 11 implies a polynomial algorithm for finding the closure under the axioms in (5) of an arbitrary set of independency statements. The algorithm simply constructs the MUG model that represents this closure. The input list itself constitute an economical encoding of the constructed MUG which requires $O(k \cdot n)$ space and time (k is the number of triplets and n is the number of variables). Similarly, this algorithm solves the ***membership problem*** for MUG and MDAGD models. The problem is to determine if a given set of independency statements Σ logically implies a candidate independency statement σ (i.e. to determine if σ holds in every MUGD (MDAGD) model that obeys Σ). The validity of σ is verified by querying the constructed MUG which represents the closure of Σ. This algorithm also implies that given a **single statement** σ it is enough to apply the axioms (5) in order to get the closure of σ under the graphoid axioms or semi-graphoid axioms. The reason is simple; only a single graph is needed for representing one statement and this graph is automatically closed under Intersection (1.f) and Contraction (1.e) which are the remaining axioms in the graphoid definition.

However, the main importance of theorem 11 is the assurance that every semi-graphoid (and graphoid) have both a MUG and MDAG representations, because these classes of dependency models obey the axioms in (5). Two issues arise with this representation scheme: The first is algorithmic, to efficiently construct a MUG representation for semi-graphoids that are not given explicitly, but in the form of a given set of statements Σ. The second issue is the amount of space required to store the MUG of an arbitrary semi-graphoid. Unfortunately, it has been recently proven ([Verma 1987b]), that any representation scheme, as clever as possible, would require on the average, exponential amount of space to store a randomly chosen semi-graphoid. Thus, while we cannot hope to use MUGs or MDAGs as perfect-maps for an arbitrary semi-graphoid, it is still interesting to explore their effectiveness in representing semi-graphoids specified as the closure of a modest number of independency statements.

A similar algorithmic problem of finding MUG representations is addressed in [Paz 1987]. Paz defines a class of dependency models, called *pseudo-graphoids*, which consists of dependency models that obey the axioms of graphoids, excluding Contraction (i.e., Symmetry, Decomposition, Weak union and Intersection). His algorithm produces a MUG representation for an arbitrary pseudo-graphoid M, by adding undirected graphs to an initial MUG until all independencies of M are represented and a perfect-map is achieved. The algorithm has the desired property that in every step the resulting MUG is an I-map of M, and in every step a better approximation of a perfect-map is obtained.

Unfortunately, a similar algorithm for generating a MUG representation for semi-graphoids has not yet been found, though some initial suggestions are given in [Paz 1987]. If found, it would render MUG dependency models well suited to represent information about relevancies and dependencies.

The use of MDAGs has not been discussed in [Paz 1987] and it remains for future research to inquire how MDAGs and MUGs can be incorporated to create more compact graphical representations.

7. CONCLUSIONS

In this thesis we have presented the concepts of completeness and weak completeness. We have established the following results:

1) There exists no finite complete set of Horn axioms for UGD models.
2) There exists no finite weakly complete (nor complete) set of Horn axioms for DAGD models.
3) There exists no finite complete set of functional-restricted axioms for DAGD models.
4) Multi-graphs dependency models (MUGD and MDAGD) have a complete axiomatization which consists of the following axioms: Symmetry, Decomposition and Weak-Union.

Related problems which remain open are:

1) The existence of a finite weakly complete set of Horn axioms for UGD models.
2) The existence of a finite weakly complete set of disjunctive axioms for DAGD models.

The first two results can be classified as "negative", since the existence of a complete set of Horn axioms in relational database has lead in the past to important results. In [Beeri & Fagin 1977], functional dependencies (FD) and Multi Valued Dependencies (MVD) were given a complete axiomatization that consists solely of Horn axioms. This result enabled Beeri ([Beeri 1980]) to solve a problem known as the *membership problem*, i.e., to find whether a candidate functional dependency or a multi valued dependency is implied by a given set of FDs and MVDs. A similar result would, clearly, be important for DAG dependency models, i.e., to verify whether a single independency statement is implied by a given set of independency statements. This paragraph also explains the importance of the first open question, namely, finding a finite weakly complete set of

Horn axioms for UGD models.

The similarities between relational database theory and dependency models theory were intensively used in this thesis. The proof of theorem 7 is similar to a construction found in [Parker and Parsaye 1980]. Lemma 9, due to Fagin, was found useful in clarifying the difficulties we encountered trying to prove the incompleteness conjecture (section 5). These similarities motivate a promising search for a more general setting in which independency statements are replaced by *constraints* (i.e., constraints are not limited to triplets and might, for example, be FDs or MVDs which are characterized by a pair of parameters rather then by three parameters) and dependency models are replaced with *constraints models* that assign truth values to the constraints (for example, a database that assigns truth values to functional dependencies). In such a setting results from database theory, even those involving dependencies different from our three-place independency statements, could be stated side by side with results from dependency models theory, and by that, unify the two seemingly different fields. The results of this thesis indicate that this is a feasible and a worthwhile effort.

The completeness theorem of section 6 (the forth result) assures that MUGD models and MDAGD models are able to represent an arbitrary semi-graphoid or graphoid. In particular this means that the most important dependency model for databases, EMVD, is fully representable by a collection of graphs, directed as well as undirected. It is surprising that database researchers have not used these graphical representations as a theoretical or a practical tool. This result motivates a future search for an algorithm that would efficiently generate multi-graph representations for semi-graphoids. Suggestions for such an algorithm are given in [Paz 1987] (without a full proof) and our result assures that the task addressed by Paz is a feasible one.

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GLOSSARY

Dependency model: A three place predicate $I(x, z, y)$ defined over disjoint subsets of elements of a fixed finite universe. $I(x, z, y)$ stands for "x is irrelevant to y, once z is known".

Graphoid: A dependency model that satisfies the following five axioms: Symmetry, Decomposition, Intersection, Weak union and Contraction.

Semi-Graphoid: A dependency model that satisfies the following four axioms: Symmetry, Decomposition, Weak union and Contraction.

Pseudo-Graphoid: A dependency model that satisfies the following four axioms: Symmetry, Decomposition, Weak-union and Intersection.

Probabilistic dependency model: A dependency model that is defined in terms of a probability distribution P.

$$I(x, z, y) \iff P(x, y \mid z) = P(x \mid z) \cdot P(y \mid z)$$

EMVD model: An *Embedded Multi Valued Dependency* model is defined in terms of a database R. $I(x, z, y)$ holds in R iff whenever the tuple $<x_1, y_1, z_1>$ and the tuple $<x_2, y_2, z_1>$ appear in the database R then the tuple $<x_1, y_2, z_1>$ appears as well (x_i, y_i and z_i are instatiations of x, y and z respectively).

Dependency Graph: A dependency model that is defined in terms of an Undirected Graph (UG) or a Directed Acyclic Graph (DAG).

UGD model: An *Undirected Graph Dependency* model (UGD) M_G is defined in terms of an undirected graph G. If x, y and z are three disjoint subsets of nodes in G then $I(x, z, y)_G$ iff every path between nodes in x and y contains at least one node in z. In other words, z is a cutset separating x from y.

DAGD model: A *Directed Acyclic Graph Dependency* model (DAGD) M_D is defined in terms of a directed acyclic graph (DAG) D. If x, y and z are three disjoint subsets of nodes in D, then by definition $I(x, z, y)_D$ iff there is no bi-directed path from a node in x to a node in y along which every node with converging arrows either is or has a descendent in z and every other node is outside z.

Dependency, independency and perfect maps: An undirected graph G is a *dependency map* (D-map) of a dependency model M over variables U if there is a one-to-one correspondence between the elements of U and the nodes of G, such that for all disjoint subsets, x, y, z, of elements we have:

$$I(x, z, y)_M \Rightarrow I(x, z, y)_G$$

Similarly, G is an *Independency map* (I-map) of M if:

$$I(x, z, y)_M \Leftarrow I(x, z, y)_G$$

G said to be a *perfect map* of M if it is both a D-map and I-map.

Independency statement a statement of the form $I(x, z, y)$.

Sound axiom: a *disjunctive axiom*

$$I(x_1, y_1, z_1) \ \& \ \cdots \ \& \ I(x_n, y_n, z_n) \Rightarrow I(\hat{x}_1, \hat{y}_1, \hat{z}_1) \ or \ \cdots \ or \ I(\hat{x}_m, \hat{y}_m, \hat{z}_m)$$

is *sound* for a class of dependency models **M** if every model in **M** that obeys the antecedents of the axiom also obeys at least one of the statements of the disjunction on the right-hand-side of the implication. When $m=1$ the axiom is said to be a *Horn axiom.*

Functional-restricted axiom: An axiom

$$I(x_1, x_2, x_3) \ \& \ \cdots \ I(x_{n-1}, x_{n-1}, x_n) => I(y_1, y_2, y_3) \ or \ \cdots \ I(y_{m-1}, y_{m-1}, y_m)$$

is *functional-restricted* if every set y_i is the result of applying the set-functions: union, intersection and negation on the x_j's.

Weak Completeness: A set of axioms **A** is *weakly complete if* for every set of statements Σ closed under **A** and for every $\sigma \notin \Sigma$ there exists a dependency model $M_\sigma \in$ **M** that obeys all statements in Σ but does not obey σ (**M** is a class of dependency models).

Completeness: A set of axioms **A** is *complete if* for every set of statements Σ closed under a set of sound axioms **A** there exists a dependency model $M \in$ **M** that obeys exactly the statements in Σ (**M** is a class of dependency models).