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GRAPHOIDS: GRAPH-BASED LOGIC FOR REASONING ABOUT RELEVANCE RELATIONS *
or
When would x tell you more about y if you already know z?

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ABSTRACT: We consider 3-place relations $I(x, z, y)$ where, x, y, and z are three non-intersecting sets of elements (e.g., propositions), and $I(x, z, y)$ stands for the statement: "Knowing z renders x irrelevant to y." We give sufficient conditions on I for the existence of a (minimal) graph G such that $I(x, z, y)$ can be validated by testing whether z separates x from y in G. These conditions define a GRAPHOID. The theory of graphoids uncovers the axiomatic basis of information relevance (e.g., probabilistic dependencies) and ties it to vertex-separation conditions in graphs. The defining axioms can also be viewed as inference rules for deducing which propositions are relevant to each other, given a certain state of knowledge.

1. INTRODUCTION

Any system that reasons about knowledge and beliefs must make use of information about relevancies. If we have acquired a body of knowledge z and now wish to assess the truth of proposition x, it is important to know whether it would be worthwhile to consult another proposition y, which is not in z. In other words, before we consult y we need to know if its truth value can potentially generate new information relative to x, information not available from z. For example, in trying to predict whether I am going to be late for a meeting, it is normally a good idea to ask somebody on the street for the time. However, once I establish the precise time by listening to the radio, asking people for the time becomes superfluous and their responses would be irrelevant. Similarly, knowing the color of X's car normally tells me nothing about the color of Y's. However, if X were to tell me that he almost mistook Y's car for his own, the two pieces of information become relevant to each other. What logic would facilitate this type of reasoning?

In probability theory, the notion of relevance is given precise quantitative underpinning using the device of conditional independence. A variable x is said to be independent of y given the information z if

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

However, it is rather unreasonable to expect people or machines to resort to numerical verification of equalities in order to extract relevance information. The ease and conviction with which people detect relevance relationships strongly suggest that such information is readily available from the organizational structure of human memory, not from numerical values assigned to its components. Accordingly, it would be interesting to explore how assertions about relevance can be tested in various models of memory and, in particular, whether such assertions can be derived by simple manipulations on graphs.

Graphs offer useful representations for a variety of phenomena. They give vivid visual display for the essential relations in the phenomenon and provide a convenient medium for people to communicate and reason about it. Graph-related concepts are so entrenched in our language that one wonders whether peo-

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ple can in fact reason any other way, except by tracing links and arrows and paths in some mental representation of concepts and relations. Therefore, if we aspire to use non-numeric logic to mimic human reasoning about knowledge and beliefs, we should make sure that most derivational steps in that logic correspond to simple operations on some graphs.

When we deal with a phenomenon where the notion of neighborhood or connectedness is explicit (e.g., family relations, electronic circuits, communication networks, etc.) we have no problem configuring a graph which represents the main features of the phenomenon. However, in modelling conceptual relations such as causation, association and relevance, it is often hard to distinguish direct neighbors from indirect neighbors; so, the task of constructing a graph representation then becomes more delicate.

This paper studies the feasibility of devising graphoid representations for relational structures in which the notion of neighborhood is not specified in advance. Rather, what is given explicitly is the relation of "in betweenness." In other words, we are given the means to test whether any given subset S of elements *intervenes* in a relation between elements x and y, but it remains up to us to decide how to connect the elements together in a graph that accounts for these interventions.

The notion of conditional independence in probability theory is a perfect example of such a relational structure. For a given probability distribution P and any three variables x, y, z, while it is fairly easy to verify whether knowing z renders x independent of y, P does not dictate which variables should be regarded as direct neighbors. Thus, many topologies might be used to display the dependencies embodied in P.

The theory of graphoids establishes a clear correspondence between probabilistic dependencies and graph representation. It tells us how to construct a unique edge-minimum graph G such that each time we observe a vertex x separated from y by a subset S of vertices, we can be guaranteed that variables x and y are independent given the values of the variables in S. Moreover, the set of neighbors assigned by G to each x coincides exactly with the boundary of x, i.e., the smallest set of variables needed to shield x from the influence of all other variables in the system. This construction is further extended by the theory of graphoids to cases where the notion of independence is not given probabilistically or numerically. We now ask what *logical* conditions should constrain the relationship: $I(x, z, y) =$ "knowing z renders x irrelevant to y" so that we can validate it by testing whether z separates x from y in some graph G. We show that two main conditions (together with symmetry and subset closure) are sufficient:

weak closure for intersection: $I(x, z \cup w, y)\ \&\ I(x, z \cup y, w) \Rightarrow I(x, z, y \cup w)$ (1)

weak closure for union: $I(x, z, y \cup w) \Rightarrow I(x, z \cup w, y)$. (2)

Loosely speaking, (1) states that if y does not affect x when w is held constant and if, simultaneously, w does not affect x when y is held constant, then neither w nor y can affect x. (2) states that learning an irrelevant fact (w) cannot help another irrelevant fact (y) become relevant. Condition (1) is sufficient to guarantee a unique construction of an edge-minimum graph G that validates $I(x, z, y)$ by vertex separation. Condition (2) guarantees that the neighborhoods defined by the edges of G coincide with the relevance boundaries defined by I. These two conditions are chosen as the defining axioms of graphoids, and are shown to account for the graphical properties of probabilistic dependencies.

This paper is organized as follows: In Section 2 we exemplify a graphoid system using probabilistic dependencies and their graphical representations. Section 3 introduces an axiomatic definition of graphoids, and states (without proofs) their graph-representation properties; the proofs can be found in [Pearl and Paz 1985]. Section 4 discusses a few extensions and outlines open problems.

2. PROBABILISTIC DEPENDENCIES AND THEIR GRAPHICAL REPRESENTATION

Let $U = \{\alpha, \beta, \cdots\}$ be a finite set of discrete-valued random variables characterized by a joint probability function $P(\cdot)$, and let x, y, and z stand for any three subsets of variables in U. We say that x and y are conditionally independent given z if

$$P(x,y \mid z) = P(x \mid z)\,P(y \mid z) \qquad \text{when } P(z) > 0 \tag{3}$$

Eq.(3) is a terse notation for the assertion that for any instantiation z_k of the variables in z and for any instantiation x_i and y_j of x and y, we have

$$P(x{=}x_i \text{ and } y{=}y_j \mid z{=}z_k) = P(x{=}x_i \mid z{=}z_k)\,P(y{=}y_j \mid z{=}z_k) \tag{4}$$

The requirement $P(z) > 0$ guarantees that all the conditional probabilities are well defined, and we shall henceforth assume that $P > 0$ for any instantiation of the variables in U. This rules out logical and functional dependencies among the variables a case which would require special treatment.

We shall use $(x \perp z \perp y)_P$ or simply $(x \perp z \perp y)$ to denote the independence of x and y given z. Thus,

$$(x \perp z \perp y)_P \iff P(x,y \mid z) = P(x \mid z)P(y \mid z) \iff P(x \mid y,z) = P(x \mid z) \tag{5}$$

Note that $(x \perp z \perp y)$ implies the conditional independence of all pairs of variables $\alpha \in x$ and $\beta \in y$, but the converse is not necessarily true.

The relation $(x \perp z \perp y)$ satisfies the following logical independent properties:

Symmetry:	$(x \perp z \perp y) \iff (y \perp z \perp x)$	(6.a)
Closure for Subsets:	$(x \perp z \perp y,w) \implies (x \perp z \perp y) \;\&\; (x \perp z \perp w)$	(6.b)
Weak Closure for Intersection:	$(x \perp y,z \perp w) \;\&\; (x \perp y,w \perp z) \implies (x \perp y \perp z,w)$	(6.c)
Weak Closure for Union:	$(x \perp y \perp z,w) \implies (x \perp y,z \perp w)$	(6.d)
Contraction:	$(x \perp y,z \perp w) \;\&\; (x \perp y \perp z) \implies (x \perp y \perp z,w)$	(6.e)

While the properties in (5) characterize the numeric representation of P, those in (6) are purely logical, void of any association with numerical forms and can be viewed, therefore, as an axiomatic definition of conditional independence. A graphical interpretation for properties (6.c) through (6.e) can be obtained by envisioning the chain x—y—z—w and associating the triplet $(x \perp z \perp y)$ with the statement "z separates x from y" or "z intervenes between x and y."

Ideally, dependent variables should be displayed as connected nodes in some graph G and independent variables as unconnected nodes. We would also like to require that if the removal of some subset S of nodes from the graph renders nodes x and y disconnected, written $<x \mid S \mid y>_G$, then this separation should correspond to conditional independence between x and y given S, namely, $<x \mid S \mid y>_G \implies (x \perp S \perp y)_P$ and conversely, $(x \perp S \perp y)_P \implies <x \mid S \mid y>_G$.

This would provide a clear graphical representation for the notion that x does not affect y directly, that its influence is mediated by the variables in S. Unfortunately, we shall next see that these two requirements might be incompatible; there might exist no way to display all the dependencies and independencies embodied in P by vertex separation in a graph.

Definition: An undirected graph G is a *dependency map* (D-map) of P if there is a one-to-one correspondence between the variables in P and the nodes of G, such that for all non-intersecting subsets, x, y, S of variables we have:

$$(x \perp S \perp y)_P \implies <x \mid S \mid y>_G \tag{7}$$

Similarly, G is an *Independency map* (I-map) of P if: $(x \perp S \perp y)_P \impliedby <x \mid S \mid y>_G$ (8)

A D-map guarantees that vertices found to be connected are indeed dependent; however, it may occasionally display dependent variables as separated vertices. An I-map works the opposite way: it guarantees that vertices found to be separated always correspond to genuinely independent variables but does not guarantee that all those shown to be connected are in fact dependent. Empty graphs are trivial D-maps, while complete graphs are trivial I-maps.

Given an arbitrary graph G, the theory of *Markov Fields* [Lauritzen 1982] tells us how to construct a probabilistic model P for which G is both a D-map and an I-map. We now ask whether the converse construction is possible.

Lemma: There are probability distributions for which no graph can be both a D-map and an I-map.

Proof: Graph separation always satisfies $<x \mid S_1 \mid y>_G \Longrightarrow <x \mid S_1 \cup S_2 \mid y>_G$ for any two subsets S_1 and S_2 of vertices. Some P's, however, may induce both $(x \perp S_1 \perp y)_P$ and NOT $(x \mid S_1 \cup S_2 \perp y)_P$. Such P's cannot have a graph representation which is both an I-map and a D-map because D-mapness forces G to display S_1 as a cutset separating x and y, while I-mapness prevents $S_1 \cup S_2$ from separating x and y. No graph can satisfy these two requirements simultaneously. Q.E.D.

An example illustrating the conditions of the proof is an experiment with two coins and a bell that rings whenever the outcomes of the two coins are the same. If we ignore the bell, the coin outcomes are mutually independent, i.e., $S_1 = \varnothing$. However, if we notice the bell (S_2), then learning the outcome of one coin should change our opinion about the other coin.

Being unable to provide a graphical description for *all* independencies, we settle for the following compromise: we will consider only I-maps but will insist that the graphs in those maps capture as many of P's independencies as possible, i.e., they should contain no superfluous edges.

Definition: A graph G is a *minimal* I-map of P if no edge of G can be deleted without destroying its I-mapness.

Theorem 1: Every P has a (unique) minimal I-map G_0 (called the *MARKOV-NET* of P) constructed by connecting *only* pairs (α, β) for which

$$(\alpha \perp U-\alpha-\beta \perp \beta)_P \;\; is \;\; FALSE \tag{9}$$

(i.e., deleting from the complete graph *all* edges (α, β) for which $(\alpha \perp U-\alpha-\beta \perp \beta)_P$).

Definition: A *Markov boundary* $B_P(\alpha)$ of variable α is a minimal subset S that renders α independent of all other variables, i.e.,

$$(\alpha \perp S \perp U-S-\alpha)_P, \; \alpha \notin S \tag{10}$$

and simultaneously, no proper subset S' of S satisfies $(\alpha \perp S' \mid U-S'-\alpha)_P$. If no S satisfies (10), define $B_P(\alpha) = U - \alpha$.

Theorem 2: Each variable α has a unique Markov boundary $B_P(\alpha)$ that coincides with the set of vertices $B_{G_0}(\alpha)$ adjacent to α in the Markov net G_0.

The usefulness of Theorem 2 lies in the fact that in many cases it is the Markov boundaries $B_P(\alpha)$ that define the organizational structure of human memory. People find it natural to identify the immediate consequences and/or justifications of each action or event, and these relationships constitute the neighborhood semantics for inference nets used in expert systems [Duda et al. 1976]. The fact that $B_P(\alpha)$ coincides with $B_{G_0}(\alpha)$ guarantees that many independencies can be validated by tests for graph separation at the knowledge level itself [Pearl 1985].

3. GRAPHOIDS

Definition: A *graphoid* is a set I of triplets (x, z, y) where x, z, y are three non-intersecting subsets of elements drawn from a finite collection $U = \{\alpha, \beta, \cdots\}$, having the following four properties. (We shall write $I(x, y, z)$ to state that the triplet (x, y, z) belongs to graphoid I.)

Symmetry	$I(x,z,y) \Longleftrightarrow I(y,z,x)$	(11.a)
Subset Closure	$I(x,z,y \cup w) \Longrightarrow I(x,z,y)\ \&\ (x,z,w)$	(11.b)
Intersection	$I(x,z \cup w,y)\ \&\ I(x,z \cup y,w) \Longrightarrow I(x,z,y \cup w)$	(11.c)
Union	$I(x,z,y \cup w) \Longrightarrow I(x,z \cup w,y)$	(11.d)

For technical convenience we shall adopt the convention that I contains all triplets in which either x or y are empty, i.e., $I(x, z, \emptyset)$.

If U stands for the set of vertices in some graph G, and if we equate $I(x, z, y)$ with the statement: "z separates between x and y," written $<x \mid z \mid y>_G$, then the conditions in (11) are clearly satisfied. However, not all properties of graph separation are required for graphoids. For example, in graphs we always have $[\, <\alpha \mid z \mid \beta>_G\ \&\ <\alpha \mid z \mid \gamma>_G \,]$ iff $<\alpha \mid z \mid \beta \cup \gamma>_G$ while property (11.b) requires only the "if" part. Similarly, graph separation dictates $<x \mid z \mid y>_G \Longrightarrow <x \mid z \cup w \mid y>_G$, $\forall w$, while (11.d) severely restricts the conditions under which a separating set z can be enlarged by w.

Definition: A graph G is said to be an *I-map* of I if there is a one-to-one correspondence between the elements in U and the vertices of G, such that, for all non-intersecting subsets x, y, S we have:

$$<x \mid S \mid y>_G \Longrightarrow I(x, S, y) \tag{12}$$

Theorem 3: Every graphoid I has a unique edge-minimum I-map G_0. $G_0 = (U, E_0)$ is constructed by connecting *only* pairs (α, β) for which the triplet $(\alpha, U-\alpha-\beta, \beta)$ is not in I, i.e.,

$$(\alpha, \beta) \notin E_0 \quad \textit{iff} \quad I(\alpha, U-\alpha-\beta, \beta) \tag{13}$$

Definition: A *relevance sphere* $R_I(\alpha)$ of an element $\alpha \in U$ is any subset S of elements for which

$$I(\alpha, S, U-S-\alpha) \text{ and } \alpha \notin S \tag{14}$$

Let $R_I^*(\alpha)$ stand for the set of all relevance spheres of α. A set is called a *relevance boundary* of α, denoted $B_I(\alpha)$, if it is in $R_I^*(\alpha)$ and if, in addition, none of its proper subsets is in $R_I^*(\alpha)$.

$B_I(\alpha)$ is to be interpreted as the smallest set that "shields" α from the influence of all other elements. Note that $R_I^*(\alpha)$ is non-empty because $I(x, z, \emptyset)$ guarantees that the set $S = U-\alpha$ satisfies (14).

Theorem 4: Every element $\alpha \in U$ in a graphoid I has a unique *relevance boundary* $B_I(\alpha)$. $B_I(\alpha)$ coincides with the set of vertices $B_{G_0}(\alpha)$ adjacent to α in the minimal graph G_0.

Corollary 1: The set of relevance boundaries $B_I(\alpha)$ forms a *neighbor system*, i.e., a collection $B_I^* = \{B_I(\alpha) : \alpha \in U\}$ of subsets of U such that (i) $\alpha \notin B_I(\alpha)$, and (ii) $\alpha \in B_I(\beta)$ iff $\beta \in B_I(\alpha)$, $\alpha, \beta \in U$

Corollary 2: The edge-minimum I-map G_0 can be constructed by connecting each α to all members of its relevance boundary $B_I(\alpha)$.

Thus we see that the major graphical properties of probabilistic dependencies are consequences of the intersection and union properties, (11.c) and (11.d), and will therefore be shared by all graphoids.

4. SPECIAL GRAPHOIDS AND OPEN PROBLEMS

4.1 Graph-induced Graphoids

The most restricted type of graphoid is that which is isomorphic to some underlying graph, i.e., *all* triplets (x, z, y) in I reflect vertex-separation conditions in an actual graph.

Definition: A graphoid I is said to be *graph-induced* if there exists a graph G such that

$$I(x, z, y) \Longleftrightarrow \langle x \mid z \mid y \rangle_G \tag{15}$$

Theorem 5: A necessary and sufficient condition for a graphoid I to be graph induced is that it satisfies the following five independent axioms:

$$I(x, z, y) \Longleftrightarrow I(y, z, x) \qquad \text{(symmetry)} \tag{16.a}$$
$$I(x, z, y \cup w) \Longrightarrow I(x, z, y) \;\&\; I(x, z, w) \qquad \text{(subset closure)} \tag{16.b}$$
$$I(x, z \cup w, y) \;\&\; I(x, z \cup y, w) \Longrightarrow I(x, z, y \cup w) \qquad \text{(intersection)} \tag{16.c}$$
$$I(x, z, y) \Longrightarrow I(x, z \cup w, y) \quad \forall w \subset U \qquad \text{(strong union)} \tag{16.d}$$
$$I(x, z, y) \Longrightarrow I(x, z, \gamma) \text{ or } I(\gamma, z, y) \quad \forall \gamma \notin x \cup z \cup y \qquad \text{(transitivity)} \tag{16.e}$$

Remarks: (16.c) and (16.d) imply the converse of (16.b), The union axiom (16.d) is unconditional and therefore stronger than the one required for general graphoids (11.d). It allows us to construct G_0 by simply deleting from a complete graph every edge (α, β) for which a triplet of the form (α, S, β) appears in I.

4.2 Probabilistic Graphoids

Definition: A graphoid is called *probabilistic* if there exists a probability distribution P on the variables in U such that $I(x, z, y)$ iff x is independent of y given z, i.e.,

$$I(x, z, y) \Longleftrightarrow (x \perp z \perp y)_P \tag{17}$$

In other words, probabilistic graphoids capture the notion of conditional independence in Probability Theory (see Section 2).

Theorem 6: Every graph-induced graphoid is probabilistic.

Since every probabilistic-independence relation satisfies (6.a)-(6.e), a necessary condition for a graphoid to be probabilistic is that, in addition to (11), it also satisfies the contraction property (6.e), i.e.,

$$I(x, y \cup z, w) \quad \& \quad I(x, y, z) \Longrightarrow I(x, y, z \cup w) \tag{18}$$

(18) can be interpreted to state that if we judge w to be irrelevant (to x) after learning some irrelevant facts z, then w must have been irrelevant before learning z. Together with the union property (11.d) it means that learning irrelevant facts should not alter the relevance status of other propositions in the system; whatever was relevant remains relevant and what was irrelevant remains irrelevant.

Conjecture: The contraction property (18) is sufficient for a graphoid to be probabilistic.

Unlike the sufficiency condition for graph-induced graphoids, we found no way of constructing a distribution P that yields $I(x, z, y) \Longrightarrow (x \perp z \perp y)_P$ for every I that satisfies (18).

4.3 Correlational Graphoids

Let U consist of n random variables $u_1, u_2, \ldots, u_n$, and let z be a subset of U such that $|z| \leq n-2$. The *partial correlation coefficient* of u_i and u_j with respect to z, denoted $\rho_{ij \cdot z}$, measures the correlation between u_i and u_j after subtracting from them the best linear estimates using the variables in z (Cramér, 1946). In other words, $\rho_{ij \cdot z}$ measures the correlation that remains after removal of any part of the variation due to the influence of the variables in z.

Definition: Let x, y, z be three nonintersecting subsets of U. A relation $I_c(x, y, z)$ is said to be *correlation-based* if for every $u_i \in x$ and $u_j \in y$ we have:

$$I_c(x, z, y) \iff \rho_{ij \cdot z} = 0 \quad (19)$$

In other words, x is considered irrelevant to y relative to z if every variable in x is uncorrelated with every variable in y, after removing the (linear) influence of the variables in z.

Theorem 7: Every correlation-based relation is a graphoid which, in addition to axioms (11), also satisfies the contraction property (18) and the converse of (11.b), i.e.,

$$I(x, z, y) \text{ and } I(x, z, w) \Longrightarrow I(x, z, y \cup w) \quad (20)$$

Conjecture: Every graphoid satisfying (18) and (20) is isomorphic to some correlation-based relation.

5. CONCLUSIONS

We have shown that the essential qualities characterizing the probabilistic notion of conditional independence are captured by two logical axioms: weak closure for intersection (6.c), and weak closure for union (6.d). These two axioms enable us to construct an edge-minimum graph in which every cutset corresponds to a genuine independence condition, and these two axioms were chosen therefore as the logical basis for graphoid systems — a more general, nonprobabilistic formalism of relevance. Vertex separation in graphs, probabilistic independence and partial uncorrelatedness are special cases of graphoid systems where the two defining axioms are augmented with additional requirements.

The graphical properties associated with graphoid systems offer an effective inference mechanism for deducing, in any given state of knowledge, which propositional variables are relevant to each other. If we identify the relevance boundaries associated with each proposition in the system, and treat them as neighborhood relations defining a graph G_0, then we can correctly deduce irrelevance relationships by testing whether the set of currently known propositions constitutes a cutset in G_0.

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