PRODUCT FORM QUEUEING NETWORKS: STATE DEPENDENCE REVISITED

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ABSTRACT

This paper examines state dependent servers and their use in product form queueing networks. State dependence in servers has been extensively investigated in the analysis of queueing systems. It is well known that several types of servers with service rates depending on the state of the queue are relatively simple to analyze in queueing networks. When the service rate depends on state attributes such as total number of customers in the queue then the service center has properties that allow it to be included in product form queueing networks. In this paper we will extend these known results by showing that it is possible to include more general forms of state dependence in service centers that can be used in product form queueing networks. These results are used to derive queueing models that can be included in product form queueing networks. Applications of these new queueing models are are illustrated in the analysis of real computing systems. Most importantly, we provide a general method for categorizing a service center as usable in a product form network. The method is based on verifying purely structural properties of a queueing system's Markov process by checking relationships between state dependent arrival and service mechanisms. Such a method is attractive because it is simple to verify and does not require solution of equilibrium state probabilities. Application of the method to several types of service centers is demonstrated.

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1 Introduction

A queueing network is a collection of service centers connected so that customers are forwarded from one center to another according to certain routing rules [Lav83]. A *product form queueing network (PFQN)* is one in which the equilibrium joint state probabilities of the network may be written as the product of the equilibrium marginal state probabilities of each service center of the network [BCM75]. These individual service centers, even when networked together, behave as independent centers. If the network consists of N service centers then the equilibrium probability of the network being in a state S is

$$P(S = S_1, S_2, \ldots, S_N) = C\, P_1(S_1) P_2(S_2) \cdots P_N(S_N)$$

where S_n is the state of service center n, and the state space of the network is the cross product of the individual servers' state spaces. The normalizing constant C is to make the individual probabilities sum to unity, given certain population constraints. The marginal state probabilities $P_n(S_n)$ are equal to the state probabilities of the isolated server, assuming the same traffic intensities.

Product form queueing networks are relatively simple to analyze. Attributes of the overall network may be evaluated by evaluating the network's individual nodes. Significant effort has gone into identifying and characterizing service centers that have properties allowing them to be included in product form queueing networks. We shall refer to a service center as a *product form service center (PFSC)* if it can be added to any PFQN and the resulting network will itself remain product form. PFSCs are the building blocks of PFQNs. Ideally we would like to say that if a service center has certain easily verifiable structural properties then it is a PFSC. This is one of our goals with respect to state dependence.

A state dependent server is one whose service rate is influenced by the state of of its queue. State dependence is valuable in evaluating real queueing systems. For example, a server may become more efficient as its queue occupancy increases. The dependence of service rate on queue state is modeled by means of a *capacity function* which specifies what fraction of the server's capacity is dedicated to satisfying a customer's service demand. Previous work has focussed on capacity functions that vary according to the number of customers in the queue [BCM75]. This paper addresses the incorporation of more general forms of state dependence into service centers of product form queueing networks. In particular, it is desired to explore forms of state dependence that allow more than just the number of customers in the queue. It is very natural to describe the state of a queueing system as the list of customers in the queue, possibly ordered by their times of arrival to the system. We will take this approach in our investigation by considering capacity functions that vary with the "natural" state of the queue. It is then interesting to ask what form such a capacity function must have in a PFSC. It will also be seen that state dependence provides a very general framework for specifying a center's service discipline.

We begin in Section 2 by reviewing relevant results that are related to the use of state dependent servers in product form queueing networks. A description of the mathematical model under study is given in Section 3. Section 4 provides the paper's major theoretical results. This section introduces the use of the path and four-cycle properties into the analysis of queueing systems. Queueing systems possessing these properties are shown to be PFSCs. The results of Section 3 are used to extend the class of known PFSCs to include virtually any kind of last-come-first-served preemptive resume (LCFSPR) center, and a modified form of the M/M/*K* center, the so called heterogeneous multiple servers (HMS) center. Section 5 will discuss applications of the results to modeling real systems. The paper is concluded in Section 6.

2 Background

The product form property, in its most basic form, was first articulated by Jackson [Jac57], who discovered that an open network of M/M/m queues[1] using the first-come-first-served (FCFS) discipline has the product form property. The class of product form networks was expanded by Gordon and Newell [GoN67] to include closed networks in which the population of customers remains constant. The work of [BCM75] was a major step toward unifying previous work and significantly extending the types of service centers that are PFSCs. This milestone paper introduced into product form queueing networks the use of multiple customer classes, mixed customer populations, new service disciplines, nonexponential service demands, and state dependent servers.[2] The work of [Mun72, CHT77, Noe79, ChM83] further advances the theory by presenting powerful models of servers and providing abstract criteria for deciding when service centers are PFSCs.

Previous research has sought to characterize PFSCs in terms of observable properties that these centers may have when viewed as isolated systems. Most generally, Muntz [Mun72] has pointed out the importance of the departure process of the isolated service center. If a center has the property that the departure process of each customer class is Poisson when the class' arrival process is Poisson, it is said to have the *M ⇒ M (Markov implies Markov)* property. *Local balance*, *station balance* [CHT77], and the *balanced* property [ChM83] are defined in terms of the specific state description of the isolated center. It is also important to mention *reversibility*, since reversibility of the underlying process of a center with a Poisson arrival stream may be used to establish that the center has the M ⇒ M property [Rei57, Kel79]. All of these properties lead to PFSCs.

[1]That is not to say that the arrival process to a networked service center is Poisson — this is in fact known not to be the case [Bur56]. We suggest that the isolated center's arrival process is Markovian because this configuration produces the same state probabilities as the networked center.

[2]Also known as *load dependent servers*, especially when the service rate is a function of the number in the queue.

The so called Baskett-Chandy-Muntz-Palacios (BCMP) queueing networks form an important subclass of the product form networks [BCM75]. BCMP queueing networks consist of four types of service centers:

1. First-come-first-served (FCFS) single-class centers with exponentially distributed service demands

2. Processor sharing (PS) multi-class centers with arbitrarily distributed service demands

3. Infinite servers (IS) multi-class centers with arbitrarily distributed service demands

4. Last-come-first-served preemptive resume (LCFSPR) multi-class servers with arbitrarily distributed service demands

This list has been expanded to include other types of servers, e.g. service in random order (SIRO) single-class centers with exponentially distributed service demands [Spi79]. All of these service center types have the $M \Rightarrow M$ property.

It is well known that if servers in a BCMP network have state dependent capacity functions, the network will still be product form. The service rate may be a function of the total number of customers in the queue, the number of customers belonging to the class of the customer in service, or a combination of both. More formally, it has been shown [BCM75, Mun83] that if the capacity dedicated to serving customers of class r is given as[3]

$$f_r(n_r, n) = g_r(n_r)\, h(n) \tag{1}$$

where the number of customers in the queue is n, n_r of which are from class r, then the server has the $M \Rightarrow M$ property.

State dependent servers have proven very useful in the analysis of queueing networks via decomposition [CHW75a, CHW75b]. A complex subnetwork may be replaced by a simpler flow equivalent service center (FESC) and the resulting network will yield performance metrics that are identical to the original network. This technique has been exceptionally fruitful in the areas of parametric and approximate analysis of the network. Efficient algorithms for solving closed queueing networks with state dependent centers have been studied by [Sau81, MiM86].

[3]It is assumed that identical shares of the server's capacity are given to all class r customers being served.

3 System Model and Definitions

A queueing network consists of *service centers* that process customers' demands and then pass the customer on to another service center according to probabilistic routing rules. There are several classes of customers in the queue and they may be treated differently with respect to service demands and routing behavior. The customer classes are numbered $1, 2, \ldots, R$. Customers may also undergo class changes according to probabilistic transition rules. The routing and class change rules may be integrated into a probability matrix $(p_{ri;sj})$ whose entries specify the probability that a customer of class r departing service center i will become a customer of class s at service center j. Customers may originate from outside the network and may also depart the network. Likewise, it is possible for a customer to remain permanently in the network by endlessly circulating through various service centers. For a formal description of queueing networks the reader may refer to [BCM75].

When discussing an isolated service center we assume that class r customers arrive in a Poisson stream of intensity λ_r. Each class r customer has a *service requirement* that is independently chosen from an exponential distribution of mean $1/\mu_r$. Upon satisfaction of its service requirement the customer departs from the center.

The mathematical model of a service center is closely related to that proposed in [CHT77]. A service center is assumed to have a *server* and a *queue*. The queue is a buffer of unbounded size that holds customers. At any point in time the queue is described by the list of customer types corresponding to the order in which they are queued (which may or may not be the order in which they arrived). We describe the state of the queue as a string over the alphabet of symbols $\Sigma \equiv \{1, 2, \ldots, R\}$, i.e. the string $x_1 x_2 \cdots x_n$ with $x_i \in \Sigma$ describes the state of a queue containing n customers where a customer of class x_1 is at the head of the queue and one of class x_n is at the tail of the queue, as illustrated in Figure 1. This may be referred to as the *natural state* of the queue. The set Σ^* consisting of all such strings is the service center's *natural state space*. Customers arrive to the queue and are assigned a position according to an *arrival function*. A class r customer arriving to a queue in state $x_1 x_2 \cdots x_n$ assumes position i with probability $a_r(x_1 x_2 \cdots x_n, i)$ producing new state $x_1 \cdots x_{i-1} r x_i \cdots x_n$. Of course we require the probabilities to sum to unity:

$$\sum_{i=1}^{n+1} a_r(x_1 x_2 \cdots x_n, i) = 1$$

Once a customer has joined the queue it may not change its relative position in the queue.

The arrival function describes the manner in which customers queue up for service. The *capacity function* describes how service is delivered to customers in the queue. In general, service may be allocated to several customers simultaneously and the relative capacity of the server may change whenever the state changes. The server will deliver service to the ith customer at the rate of $b(x_1 x_2 \cdots x_n, i)$ when the queue is in state $x_1 x_2 \cdots x_n$, i.e. if no change of state oc-

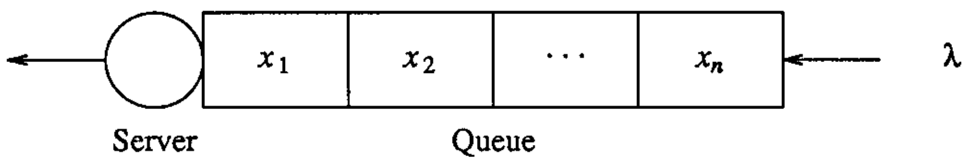


Figure 1. Service center in state $x_1 x_2 \cdots x_n$.

curs within Δt time units, the ith customer will have its remaining service requirement reduced by $b(x_1x_2 \cdots x_n, i)\ \Delta t$ time units. Thus the server is modeled as a resource of potentially unlimited capacity with the capability of discriminating among customers on the basis of class and queue position.

The arrival function specifies the queueing discipline (i.e. the way customers line up), whereas the capacity function specifies the actual service discipline (i.e. the way customer get serviced). Many interesting queueing systems can be derived by jointly specifying the capacity and arrival functions, including all of the well known BCMP service centers. The practical utility of these functions has been illustrated in [CHT77].

The natural state space and queueing function may be used to further refine the service center model. We are often interested in centers where customers are queued in the order of their arrival times. Such systems will be said to use *time ordered queueing of arrivals (TOQA).* They have an arrival function of the form

$$a_r(x_1x_2 \cdots x_n, i) = \begin{cases} 1 & \text{if } i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

for each state $x_1x_2 \cdots x_n$ and class r, i.e. all arrivals join the tail of the queue. A state $x_1x_2 \cdots x_n$ expressed in TOQA represents a queue in which x_1 arrived before x_2 which arrived before x_3, etc. Even with TOQA it is not always possible to unambiguously identify customers. For example, the state (11221211) could undergo a transition to state (1122121). It is clear that one of the last two class 1 customers have left the queue, but we can not tell which one. By associating a sequence number with the class symbol, it is possible to uniquely identify every customer in the queue. In this approach, called *time ordered queueing of arrivals with customer identification (TOQACI),* the state of the queue is written $\langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_n, m_n\rangle$ where $x_i \in \Sigma$ and m_i is a positive integer. The rules are that an arrival is placed at the tail of the queue with the next highest sequence number and departing customers are merely removed from the queue. Thus the arrival of a class r customer to the queue in state $\langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_n, m_n\rangle$ would produce state $\langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_n, m_n\rangle\langle r, m_n{+}1\rangle$ and the departure of a customer would simply result in the deletion of the customer from the list. It is clear that $m_1 < m_2 < \cdots < m_n$ when sequence numbers are managed according to these rules. We can now tell, for instance, that a transition from state $(\langle 1, 2\rangle\langle 1, 3\rangle\langle 2, 5\rangle\langle 2, 6\rangle)$ to state $(\langle 1, 2\rangle\langle 1, 3\rangle\langle 2, 6\rangle)$ occurred when the third customer (the class 2 customer with sequence number 5) departed.

To summarize, we will consider service centers with R classes of customers. Class r customers arrive to the isolated center on average every $1/\lambda_r$ time units according to a Poisson rule and require an average amount of total service of $1/\mu_r$ units chosen from an exponential distribution. The system is normally described by the natural state reflecting the queue's occupancy. The queueing and service disciplines are specified by means of arrival and capacity functions $a_r(\cdot, \cdot)$ and $b(\cdot, \cdot)$ which map elements of the natural state space onto nonnegative real numbers.

By restricting the arrival function to always place an arriving customer at the tail of the queue, we can narrow down to TOQA systems. TOQA centers with sequence numbers to uniquely identify customers yield a class of service centers known as TOQACI systems. The stochastic process of principal interest is the evolution of the natural state of the system over time. This process is a homogeneous Markov process and is called the *natural Markov process* of the isolated service center. The natural Markov process will be denoted by the family of random variables indexed over time, viz. $\Omega \equiv \{S(t) : t \geq 0\}$, where $S(t)$ is the natural state of the service center at time t.

4 Using State Dependent Servers in Product Form Networks

Given the framework presented in Section 3, it is reasonable to ask for a characterization of these service centers with respect to their inclusion in PFQNs. It is well known that a reversible Markov process can be identified by purely structural characteristics of its states and transition rates, e.g. Kolmogorov's criteria [Kel79]. Similarly, it is possible to identify certain service centers as PFSCs by simple inspection of their structural properties. We will consider various service centers, based on their queueing disciplines.

4.1 Non-TOQA PFSCs

Here we study systems that queue customers without necessarily arranging them in order of arrival. In these centers a wide range of state dependent service disciplines are possible. The form of the service function will, in general, be related to the arrival function.

When studying a service center, we are typically interested in its underlying Markov process. The stochastic process consisting of the center's natural state over time is a homogeneous Markov process. If the service center is well behaved the natural Markov process will, in general, be stationary.

Let $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m$ be a path[4] in a Markov state graph from state X_1 to state X_m via states $X_2, X_3, \ldots, X_{m-1}$. The *value* of the path is defined as

$$\prod_{i=1}^{m-1} \frac{R(X_i \rightarrow X_{i+1})}{R(X_{i+1} \rightarrow X_i)}$$

where $R(X \rightarrow Y)$ represents the transition rate from state X to state Y. The value of the path is computed by dividing the product of all forward arc labels by the product of all backward arc labels. A center is said to have the *path property* when its Markov state graph is such that given any state $x_1 x_2 \cdots x_n$, the values of all paths of length n from $\varnothing$ to $x_1 x_2 \cdots x_n$ are identical. We

[4]If a path in a state graph includes a hop from state X to state Y it is assumed that there must be a nonzero transition rate on the arcs connecting the two states.

denote this unique value by $V(x_1x_2 \cdots x_n)$. In a center with the *four-cycle property* the value of any cycle of the state graph containing four states is 1.

It is straightforward to verify whether a state graph has the path or four-cycle property. In a service center with arrival and capacity functions $a_r(\cdot, \cdot)$ and $b(\cdot, \cdot)$ there is a simple expression for the transition rate between any two states. There are essentially two cases to distinguish for $R(X \to Y)$:

1. If $X = Ur^kV$ where[5] U is a string of l symbols, the last of which is not r, and V does not begin with the symbol r, then the transition rate from X to $Y = Ur^{k+1}V$ is

$$R(X \to Y) = \lambda_r \sum_{i=l+1}^{l+k+1} a_r(X, i)$$

This type of transition is caused by class r arrivals to positions $l+1, l+2, \ldots, l+k+1$ of the queue when it is in state X. An arrival of a class r customer to any of these positions moves the queue into state Y.

2. Similarly, if X is as above and $Y = Ur^{k-1}V$, then the transition rate from X to Y is

$$R(X \to Y) = \mu_r \sum_{i=l+1}^{l+k} b(X, i)$$

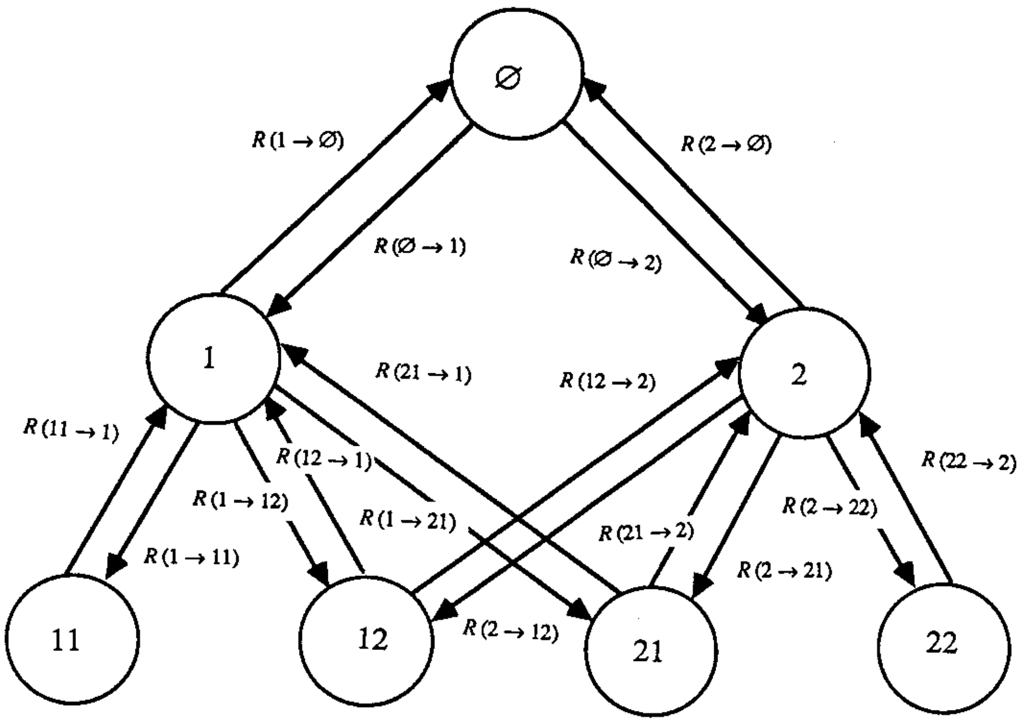
This type of transition results from the departure of a class r customer in queue positions $l+1, l+2, \ldots, l+k$ while the queue is in state X. A departure from any of these positions moves the queue into state Y.

All other transition rates are zero. This is because the only nontrivial state transitions in a service center consist of unit jumps — customer departures and arrivals. An example of a fragment of a service center's labeled state graph with $R = 2$ is illustrated in Figure 2.

Having labeled the arcs of the state graph, it is possible to check for the path or four-cycle property directly. In the case of the path property one begins with the root state $\varnothing$ and traverses the graph in a breadth-first manner. Path values at level n of the graph (i.e. states corresponding to a queue occupancy of n customers) are computed in terms of values of paths terminating at level $n-1$. Likewise, the four-cycle property means every configuration of four adjacent states, as shown in Figure 3, will have the relation

$$\frac{R(x_1x_2 \cdots x_irx_{i+1} \cdots x_jsx_{j+1} \cdots x_n \to x_1x_2 \cdots x_irx_{i+1} \cdots x_n)}{R(x_1x_2 \cdots x_irx_{i+1} \cdots x_n \to x_1x_2 \cdots x_irx_{i+1} \cdots x_jsx_{j+1} \cdots x_n)}$$

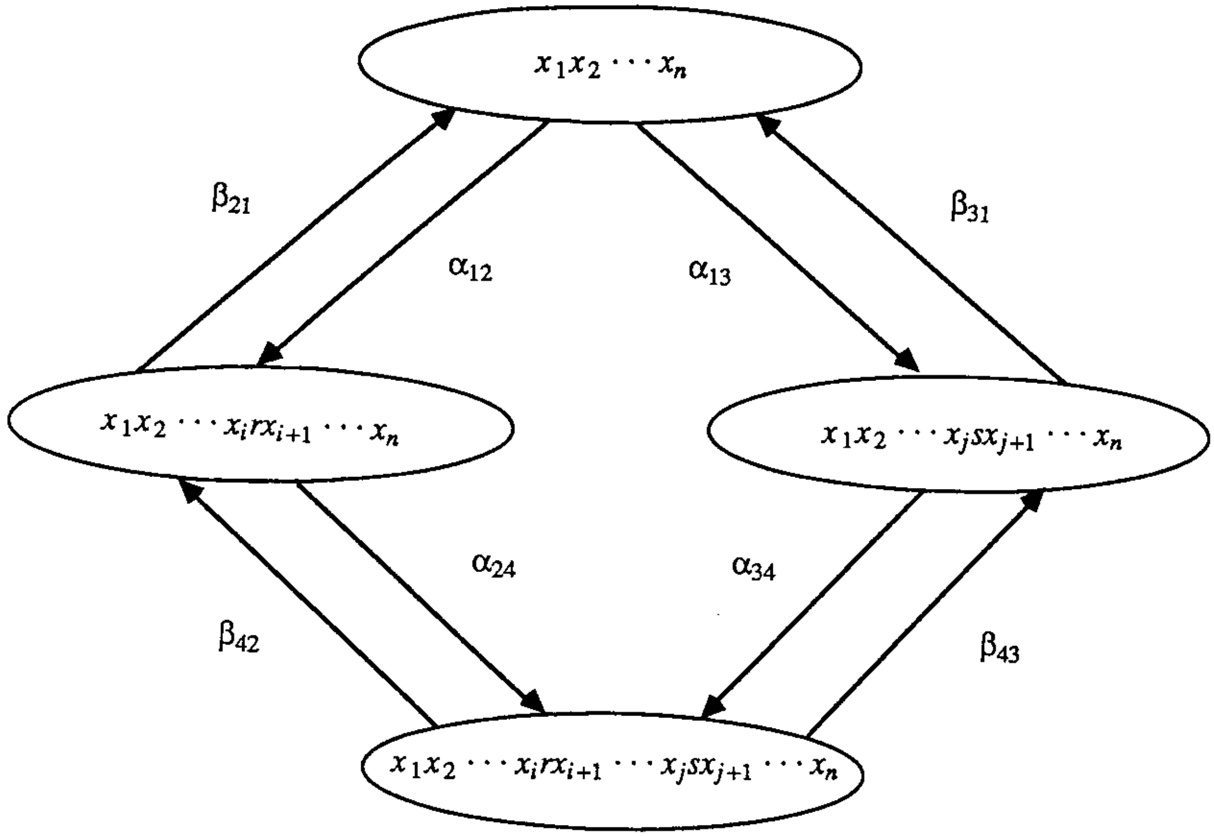
[5]We use r^k to represent a string of k occurrences of the symbol r.



$R(\varnothing \to 1) = \lambda_1 \, a_1(\varnothing, 1)$
$R(\varnothing \to 2) = \lambda_1 \, a_2(\varnothing, 1)$
$R(1 \to 11) = \lambda_1 \, [a_1(1, 1) + a_1(1, 2)]$
$R(1 \to 12) = \lambda_2 \, a_2(1, 2)$
$R(1 \to 21) = \lambda_2 \, a_2(1, 1)$
$R(2 \to 12) = \lambda_1 \, a_1(2, 1)$
$R(2 \to 21) = \lambda_1 \, a_1(2, 2)$
$R(2 \to 22) = \lambda_2 \, [a_2(2, 1) + a_2(2, 2)]$

$R(1 \to \varnothing) = \mu_1 \, b(1, 1)$
$R(2 \to \varnothing) = \mu_2 \, b(2, 1)$
$R(11 \to 1) = \mu_1 \, [b(11, 1) + b(11, 2)]$
$R(12 \to 1) = \mu_2 \, b(12, 2)$
$R(12 \to 2) = \mu_1 \, b(12, 1)$
$R(21 \to 1) = \mu_2 \, b(21, 1)$
$R(21 \to 2) = \mu_1 \, b(21, 2)$
$R(22 \to 2) = \mu_2 \, [b(22, 1) + b(22, 2)]$

Figure 2. Fragment of the natural state graph of a service center with two customer classes.



$$\alpha_{12} = R\,(x_1x_2 \cdots x_n \rightarrow x_1x_2 \cdots x_i r x_{i+1} \cdots x_n)$$
$$\alpha_{13} = R\,(x_1x_2 \cdots x_n \rightarrow x_1x_2 \cdots x_j s x_{j+1} \cdots x_n)$$
$$\alpha_{24} = R\,(x_1x_2 \cdots x_i r x_{i+1} \cdots x_n \rightarrow x_1x_2 \cdots x_i r x_{i+1} \cdots x_j s x_{j+1} \cdots x_n)$$
$$\alpha_{34} = R\,(x_1x_2 \cdots x_j s x_{j+1} \cdots x_n \rightarrow x_1x_2 \cdots x_i r x_{i+1} \cdots x_j s x_{j+1} \cdots x_n)$$

$$\beta_{21} = R\,(x_1x_2 \cdots x_i r x_{i+1} \cdots x_n \rightarrow x_1x_2 \cdots x_n)$$
$$\beta_{31} = R\,(x_1x_2 \cdots x_j s x_{j+1} \cdots x_n \rightarrow x_1x_2 \cdots x_n)$$
$$\beta_{42} = R\,(x_1x_2 \cdots x_i r x_{i+1} \cdots x_j s x_{j+1} \cdots x_n \rightarrow x_1x_2 \cdots x_i r x_{i+1} \cdots x_n)$$
$$\beta_{43} = R\,(x_1x_2 \cdots x_i r x_{i+1} \cdots x_j s x_{j+1} \cdots x_n \rightarrow x_1x_2 \cdots x_j s x_{j+1} \cdots x_n)$$

Figure 3. Diamond configuration for checking the four-cycle property.

$$\times \frac{R(x_1x_2 \cdots x_i r x_{i+1} \cdots x_n \to x_1x_2 \cdots x_n)}{R(x_1x_2 \cdots x_n \to x_1x_2 \cdots x_i r x_{i+1} \cdots x_n)}$$

$$= \frac{R(x_1x_2 \cdots x_i r x_{i+1} \cdots x_j s x_{j+1} \cdots x_n \to x_1x_2 \cdots x_j s x_{j+1} \cdots x_n)}{R(x_1x_2 \cdots x_j s x_{j+1} \cdots x_n \to x_1x_2 \cdots x_i r x_{i+1} \cdots x_j s x_{j+1} \cdots x_n)}$$

$$\times \frac{R(x_1x_2 \cdots x_j s x_{j+1} \cdots x_n \to x_1x_2 \cdots x_n)}{R(x_1x_2 \cdots x_n \to x_1x_2 \cdots x_j s x_{j+1} \cdots x_n)}$$

assuming $i < j$ and $r \neq s$. This entails examining every diamond shaped configuration of states and verifying the multiplicative relationship between the labels on the diamond's eight arcs. Other configurations, like the butterfly shape shown in Figure 4, reduce to the diamond pattern, as suggested by the figure.

The reason for wanting to know whether the center has the path or four-cycle property is that all such service centers are PFSCs. This is formally demonstrated in the next theorem and its corollary.

Theorem 1. Let $\Omega = \{S(t) : t \geq 0\}$ be the natural Markov process of a service center. Suppose that the service center has a natural state graph such that if X and Y are any two states, then $R(X \to Y)$ and $R(Y \to X)$ are both positive or are both zero. The following assertions are equivalent:

1. The center's natural Markov process Ω is reversible.

2. The center's natural state graph has the path property.

Moreover, if any of the above conditions holds, then the center is a PFSC with state probability distribution given by

$$P(X) = P(\varnothing)\, V(X) \tag{2}$$

for any state X.[6]

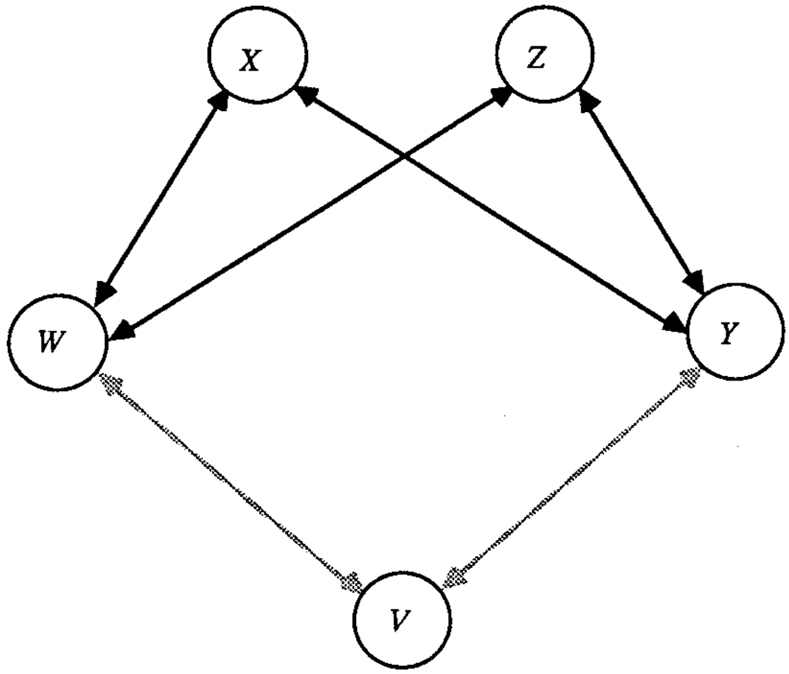
Proof. To see that (1) $\Rightarrow$ (2) we apply Kolmogorov's criteria [Kel79], which state that a stationary Markov process is reversible if and only if its transition rates satisfy

$$R(X_1 \to X_2)\, R(X_2 \to X_3) \cdots R(X_{m-1} \to X_m)\, R(X_m \to X_1)$$

$$= R(X_1 \to X_m)\, R(X_m \to X_{m-1}) \cdots R(X_3 \to X_2)\, R(X_2 \to X_1)$$

for any sequence of states $X_1, X_2, \ldots, X_m$. Let X_n be a state containing n customers and let

[6]Assume that $V(\varnothing) \equiv 1$.



$$\frac{R(X \to Y)}{R(Y \to X)} \frac{R(Y \to V)}{R(V \to Y)} \frac{R(V \to W)}{R(W \to V)} \frac{R(W \to X)}{R(X \to W)} = 1$$

$$\frac{R(Z \to Y)}{R(Y \to Z)} \frac{R(Y \to V)}{R(V \to Y)} \frac{R(V \to W)}{R(W \to V)} \frac{R(W \to Z)}{R(Z \to W)} = 1$$

$$\therefore \frac{R(X \to Y)}{R(Y \to X)} \frac{R(Y \to Z)}{R(Z \to Y)} \frac{R(Z \to W)}{R(W \to Z)} \frac{R(W \to X)}{R(X \to W)} = 1$$

Figure 4. Reduction of the butterfly configuration to the diamond configuration. If all diamond shaped four-cycles have value 1, then any four-cycle will have value 1.

$\varnothing \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$ and $\varnothing \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_{n-1} \rightarrow X_n$ be any two n-hop paths from the root state to X_n. Considering the cycle formed by traversing the first path from $\varnothing$ to X_n and then backtracking from X_n to $\varnothing$ along the second path, we obtain from Kolmogorov's criteria

$$R(\varnothing \rightarrow X_1) R(X_1 \rightarrow X_2) \cdots R(X_{n-1} \rightarrow X_n)$$

$$\times R(X_n \rightarrow Y_{n-1}) R(Y_{n-1} \rightarrow Y_{n-2}) \cdots R(Y_1 \rightarrow \varnothing)$$

$$= R(\varnothing \rightarrow Y_1) R(Y_1 \rightarrow Y_2) \cdots R(Y_{n-1} \rightarrow X_n)$$

$$\times R(X_n \rightarrow X_{n-1}) R(X_{n-1} \rightarrow X_{n-2}) \cdots R(X_1 \rightarrow \varnothing)$$

After division and rearrangement of terms we obtain

$$\frac{R(\varnothing \rightarrow X_1) R(X_1 \rightarrow X_2) \cdots R(X_{n-1} \rightarrow X_n)}{R(X_1 \rightarrow \varnothing) R(X_2 \rightarrow X_1) \cdots R(X_n \rightarrow X_{n-1})}$$

$$= \frac{R(\varnothing \rightarrow Y_1) R(Y_1 \rightarrow Y_2) \cdots R(Y_{n-1} \rightarrow X_n)}{R(Y_1 \rightarrow \varnothing) R(Y_2 \rightarrow Y_1) \cdots R(X_n \rightarrow Y_{n-1})}$$

Thus any two paths of length n from $\varnothing$ to X_n have the same value. Therefore the center's state graph has the path property.

Next the converse (2) $\Rightarrow$ (1) is proved. Suppose that a center's state graph has the path property. Let $X_1, X_2, \ldots, X_n$ be a sequence of states. If any transition rate equals zero, say $R(X_i \rightarrow X_{i+1}) = 0$, then the reverse rate must also be zero, i.e. $R(X_{i+1} \rightarrow X_i) = 0$. It is then clear that Kolmogorov's criteria hold for this sequence. Assume then that $R(X_i \rightarrow X_{i+1}) \neq 0$ and $R(X_{i+1} \rightarrow X_i) \neq 0$ for all i (modulo n). If state X_i has one less customer than state X_{i+1} then X_i is on a path from the root state to X_{i+1} and by the path property

$$V(X_{i+1}) = \frac{R(X_i \rightarrow X_{i+1})}{R(X_{i+1} \rightarrow X_i)} V(X_i)$$

Similarly if state X_i has one more customer than state X_{i+1} then

$$V(X_i) = \frac{R(X_{i+1} \rightarrow X_i)}{R(X_i \rightarrow X_{i+1})} V(X_{i+1})$$

In either case $V(X_i) R(X_i \rightarrow X_{i+1}) = V(X_{i+1}) R(X_{i+1} \rightarrow X_i)$ for all i. Multiplying these equations over all values of i (modulo n) we get

$$\prod_{i=1}^{n} V(X_i) R(X_i \rightarrow X_{i+1}) = \prod_{i=1}^{n} V(X_{i+1}) R(X_{i+1} \rightarrow X_i)$$

All the $V(X_i)$ cancel and we are left with

$$\prod_{i=1}^{n} R(X_i \to X_{i+1}) = \prod_{i=1}^{n} R(X_{i+1} \to X_i)$$

which is Kolmogorov's criteria. Since any sequence of states satisfies Kolmogorov's criteria the natural Markov process Ω is reversible.

We now demonstrate that Equation (2) is a solution for the equilibrium state probability distribution of Ω. The proof is by induction on the size of X. The basis case of $X = \varnothing$ is trivial. Assume that the form holds for all states of occupancy less that n and that state X has n customers. Let Y be a state of occupancy $n - 1$ that has a nonzero transition rate to state X. Since Ω is reversible we know [Kel79, HeS82] that

$$P(X)\,R(X \to Y) = P(Y)\,R(Y \to X) \tag{3}$$

Using the induction hypothesis we write

$$P(Y) = P(\varnothing)\,V(Y) \tag{4}$$

Substituting Equation (4) back into Equation (3) yields

$$P(X) = \frac{R(Y \to X)}{R(X \to Y)}\,P(\varnothing)\,V(Y)$$

$$= P(\varnothing)\,V(X)$$

which is Equation (2).

That the center is a PFSC follows from its reversibility in a manner that parallels the arguments of [Rei57].[7] One demonstrates the M $\Rightarrow$ M property of a multiclass center by showing that each class has a Poisson departure stream [Mun72, Mun83]. Let $\Omega^R \equiv \{S^R(t) : S^R(t) = S(\tau - t)\}$ be the reversed process of Ω. Since the process Ω starts at 0 we choose τ to be a sufficiently large quantity to ensure that the process has reached steady state. The intuitive idea is that since Ω and Ω^R are statistically indistinguishable, successive jumps in the population of class r customers are similarly distributed. These jumps in Ω^R correspond to departures from the center, while in Ω they are simply arrivals. From this we conclude that the departure process has the Poisson distribution.

Define $\kappa(X, r)$ to be the number of times the symbol r occurs in string X. Let t be a point in time and fix T. Let $t_i \equiv t + (i/k)T$ for any positive integer k and $1 \le i \le k$, and define

$$E_r \equiv \{[S(t_0), S(t_1), \ldots, S(t_k)] :$$

[7]Reich concentrated strictly on single-class, Markovian service centers.

$$\kappa[S(t_0), r] < \kappa[S(t_1), r] \geq \kappa[S(t_2), r] \geq \cdots$$

$$\cdots \geq \kappa[S(t_{k-1}), r] < \kappa[S(t_k), r]\}$$

$$E_r^R \equiv \{[S^R(t_0), S^R(t_1), \ldots, S^R(t_k)] :$$

$$\kappa[S^R(t_0), r] < \kappa[S^R(t_1), r] \geq \kappa[S^R(t_2), r] \geq \cdots$$

$$\cdots \geq \kappa[S^R(t_{k-1}), r] < \kappa[S^R(t_k), r]\}$$

Note that

$$E_r^R = \{[S(\tau - t_k), S(\tau - t_{k-1}), \ldots, S(\tau - t_0)] :$$

$$\kappa[S(\tau - t_k), r] > \kappa[S(\tau - t_{k-1}), r] \leq \kappa[S(\tau - t_{k-2}), r] \leq \cdots$$

$$\cdots \leq \kappa[S(\tau - t_1), r] > \kappa[S(\tau - t_0), r]\}$$

The event E_r may be interpreted as saying that a class r arrival occurred in $(t_0, t_1]$ and in $(t_{k-1}, t_k]$, and event E_r^R says that a class r departure occurred in $(\tau - t_k, \tau - t_{k-1}]$ and in $(\tau - t_1, \tau - t_0]$. In the limit as $k \to \infty$, E_r approximates the event of a class r arrival at time t and one at time $\tau + T$ with no intervening class r arrivals in the interval $(t, t+T)$. Similarly E_r^R tends to approximate the event of a class r customer's departure at $\tau - t - T$ and one at $\tau - t$ with no class r departures in the time interval $(\tau - t - T, \tau - t)$.

Since the arrival process is Poisson we know that the probability density of E_r approaches $\lambda_r e^{-\lambda_r T}$ as $k \to \infty$. By the reversibility of Ω we also know that $[S(t_0), S(t_1), \ldots, S(t_k)]$ and $[S^R(t_0), S^R(t_1), \ldots, S^R(t_k)]$ have identical distributions. Thus the probability density of E_r^R approaches $\lambda_r e^{-\lambda_r T}$ as $k \to \infty$. Since t and T were arbitrarily chosen, we conclude that the distribution of elapsed time between any two consecutive class r departures is exponential. If the interdeparture times are independent, exponentially distributed random variables, then the departure process is Poisson. Hence the center has the M $\Rightarrow$ M property and is a PFSC. ■

The following corollary to Theorem 1 represents the special application of Kolmogorov's criteria to queueing systems. Kolmogorov's criteria, when applied to an arbitrary Markov process, require *any* cycle to have value 1. Because transitions are restricted to arrivals and departures, conventional queueing systems have Markov processes with special attributes that allow us to equate reversibility to a much simplified version of Kolmogorov's criteria, viz. the four-cycle property. Checking only four-cycles is vastly easier than checking *all* cycles. A service center's arrival and capacity functions specify transition rates between adjacent states. It is a simple matter to verify that an arbitrary diamond pattern, as shown in Figure 3, has value 1. If the arrival and capacity functions can be defined so that the four-cycle property holds, then the service center can be shown to be M ⇒ M.

Corollary 1. Let $\Omega = \{S(t) : t \geq 0\}$ be the natural Markov process of a service center. Assume that the service center has positive arrival and capacity functions, i.e. $a_r(X, i) > 0$ and $b(X, i) > 0$ for each r, X, and i. The following are equivalent:

1. The center's natural Markov process Ω is reversible.

2. The center's natural state graph has the four-cycle property.

Proof. First we show that (1) ⇒ (2). Choose any four adjacent states X, Y, Z, and W connected in a cycle by arcs labeled with positive transition rates. Applying Kolmogorov's criteria to the four-cycle $X \to Y \to Z \to W \to X$ we get

$$R(X \to Y)\,R(Y \to Z)\,R(Z \to W)\,R(W \to X)$$

$$= R(X \to W)\,R(W \to Z)\,R(Z \to Y)\,R(Y \to X)$$

Since the transition rates on the right hand side are all positive, we may divide by the right hand side and obtain 1 as the value of the four-cycle. Thus all four-cycles have value 1 and the center has the four-cycle property.

Next we show that (2) ⇒ (1). We will show that the state graph must have the path property. The proof is by induction on the number of customers in a state. For the basis case it is clear that all states containing exactly one customer have the path property since there is only one path to the root state. Assume that, given any state with fewer than n customers, there is a unique value for every minimal length path from the root state to the given state. Consider the state

$$X \equiv x_1 x_2 \cdots x_n$$

and any two n-hop paths from the null state to it. Suppose that the penultimate states of each path are

$$Y \equiv x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$$

which is X minus the customer in position i of the queue, and

$$Z \equiv x_1 x_2 \cdots x_{j-1} x_{j+1} \cdots x_n$$

which is X minus the customer in position j of the queue. Assume that $i \leq j$. Since the arrival and capacity functions are positive everywhere the state

$$W \equiv x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n$$

has positive transition rates to and from both Y and Z. Now the four-cycle $X \to Y \to Z \to W$ has value 1. We may then write

$$\frac{R(W \to Y)}{R(Y \to W)} \times \frac{R(Y \to X)}{R(X \to Y)} = \frac{R(W \to Z)}{R(Z \to W)} \times \frac{R(Z \to X)}{R(X \to Z)} \tag{5}$$

State W, containing fewer than n customers, has a unique value $V(W)$ by the induction hypothesis. Furthermore, applying the induction hypothesis to states Y and Z we may write

$$V(Y) = \frac{R(W \to Y)}{R(Y \to W)} V(W) \tag{6}$$

and

$$V(Z) = \frac{R(W \to Z)}{R(Z \to W)} V(W) \tag{7}$$

Combining Equations (5) through (7) we obtain

$$\frac{R(Y \to X)}{R(X \to Y)} V(Y) = \frac{R(Z \to X)}{R(X \to Z)} V(Z)$$

The left hand side of this last equation is the value of the first path from the root state to X while the right hand side is the value of the second path. Hence any two paths from the root state to X must have identical values. Thus the natural Markov process Ω has the path property and by Theorem 1 it must be reversible. ■

As an illustration of the power of the four-cycle property we consider a service center consisting of K servers. The servers are numbered from 1 to K and server k has capacity μ_k. Let $\mathrm{K} \equiv \{1, 2, \ldots, K\}$ be the set of servers. When a customer arrives to the center it chooses a server randomly from the pool of idle servers. If no server is idle, the customer joins the end of the queue and waits until all preceding customers have gone into service before it can be served. This model differs from the classical M/M/K model in that it admits a heterogeneous set of servers that may have differing operating characteristics, expressed primarily through their distinct capacities. This type of service center will be called the heterogeneous multiple servers (HMS) center.

Suppose that all customer classes have the same exponential service requirement. We may assume that the mean service requirement has been normalized to 1 by suitably scaling the capacities $\mu_1, \mu_2, \ldots, \mu_K$. If we let $S \subseteq \mathrm{K}$ be the set of busy servers and n be the number of customers waiting for service at a given point in time, then (S, n) provides a description of the state of the HMS center. The state of the center over time is a stationary Markov process.

We apply the four-cycle result to the HMS center to show that is is a PFSC.

Corollary 2. Any HMS service center with identical service requirements for all classes is a PFSC. Its equilibrium state probability distribution is

$$P(S,\ n) = \frac{P(\emptyset,\ 0)}{K\,(K-1)\cdots(K-|S|+1)} \left[\prod_{k\in S} \frac{\lambda}{\mu_k}\right] \left[\frac{\lambda}{\mu}\right]^n \tag{8}$$

where $\lambda \equiv \sum_{r=1}^{R} \lambda_r$ and $\mu \equiv \sum_{k=1}^{K} \mu_k$.

Proof. The transition rates of the center's Markov process are specified as follows:

1. If $k \notin S \subset \mathsf{K}$, then

$$R\,[(S,\ 0) \to (S \cup \{k\},\ 0)] = \frac{\lambda}{K - |S|}$$

2. If $k \in S \subset \mathsf{K}$, then

$$R\,[(S,\ 0) \to (S - \{k\},\ 0)] = \mu_k$$

3. If $n = 0,\ 1,\ \ldots$ then

$$R\,[(\mathsf{K},\ n) \to (\mathsf{K},\ n+1)] = \lambda$$

4. If $n = 1,\ 2,\ \ldots$ then

$$R\,[(\mathsf{K},\ n) \to (\mathsf{K},\ n-1)] = \mu$$

All other transition rates are 0. Refer to Figure 5 for a representation of the HMS center's Markov state graph.

All four-cycles in the state graph are of the form $(S,\ 0) \to (S \cup \{k\},\ 0) \to (S \cup \{k,\ l\},\ 0) \to (S \cup \{l\},\ 0) \to (S,\ 0)$ where $S \subset \mathsf{K}$, $k,\ l \notin S$, and $k \neq l$. The value of this four-cycle is

$$\left[\frac{\lambda}{K-|S|}\ \frac{1}{\mu_k}\right] \times \left[\frac{\lambda}{K-|S|-1}\ \frac{1}{\mu_l}\right] \times \left[\mu_k\ \frac{K-|S|-1}{\lambda}\right] \times \left[\mu_l\ \frac{K-|S|}{\lambda}\right]$$

which has value 1.

The arguments in the proof of Theorem 1 and Corollary 1 are readily adapted to the new state description of the HMS center. Thus the center is a PFSC. The expression of Equation (8) is the unique value of any path from the empty state $(\emptyset,\ 0)$ to state $(S,\ n)$. ■

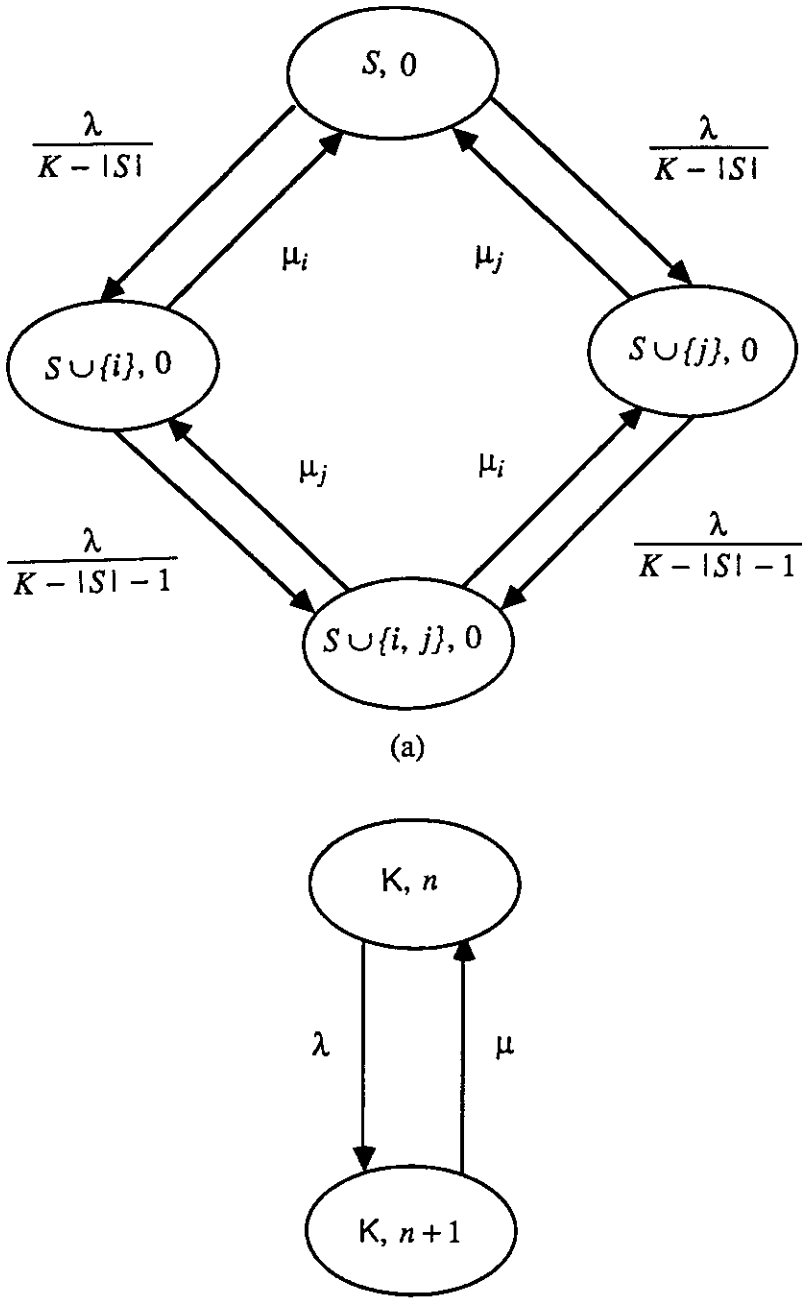


Figure 5. Fragments of the HMS state graph. (a) States for which $S \subset K$ and $i, j \notin S$. (b) States for which $n \geq 0$.

4.2 *TOQA PFSCs*

We now restrict our attention to service centers in which arriving customers always join the tail of the queue. We still allow the capacity function to be chosen in a way that reflects the manner in which service is allocated to customers by the server.

The last theorem has one immediate corollary that proves valuable in the analysis of PFQNs. In a LCFSPR service center allocation of the server's capacity may be extremely sensitive to the center's queue state. In fact, from a PFQN point of view, the form of the capacity function is irrelevant, except for the way that it affects the state probability distribution. That is, a LCFSPR center is a PFSC regardless of the form taken by its capacity function!

Corollary 3. In a LCFSPR TOQA service center the capacity function $b(\cdot, \cdot)$ may take any form and the center will be a PFSC.

Proof. We start by pointing out that a LCFSPR TOQA center is defined to have arrival and capacity functions that obey the following constraints:

$$a_r(x_1x_2 \cdots x_n, i) = \begin{cases} 1 & \text{if } i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

for each $r \in \Sigma$, and $b(x_1x_2 \cdots x_n, i) = 0$ for $1 \le i < n$. We allow $b(x_1x_2 \cdots x_n, n)$ to assume any positive value, which results in a capacity function that is completely state dependent. It may be seen that the arrival and capacity functions specified above result in a tree-structured state graph, as shown in Figure 6. Thus there is only one nontrivial path from the root state to any given state, i.e. the graph has the path property. From Theorem 1 the center is a PFSC with state probability distribution given as

$$P(x_1x_2 \cdots x_n) = P(\varnothing)\, V(x_1x_2 \cdots x_n)$$

$$= P(\varnothing) \prod_{i=1}^{n} \frac{\lambda_i}{\mu_i\, b(x_1x_2 \cdots x_i, i)}$$

■

It is natural to ask what forms the capacity function may take on in a TOQA PFSC. It will be seen shortly that the capacity function for a particular customer may depend on the total number of customers in the queue, the number of customers in the customer's class, and the customer's queue position relative to other members of its class.

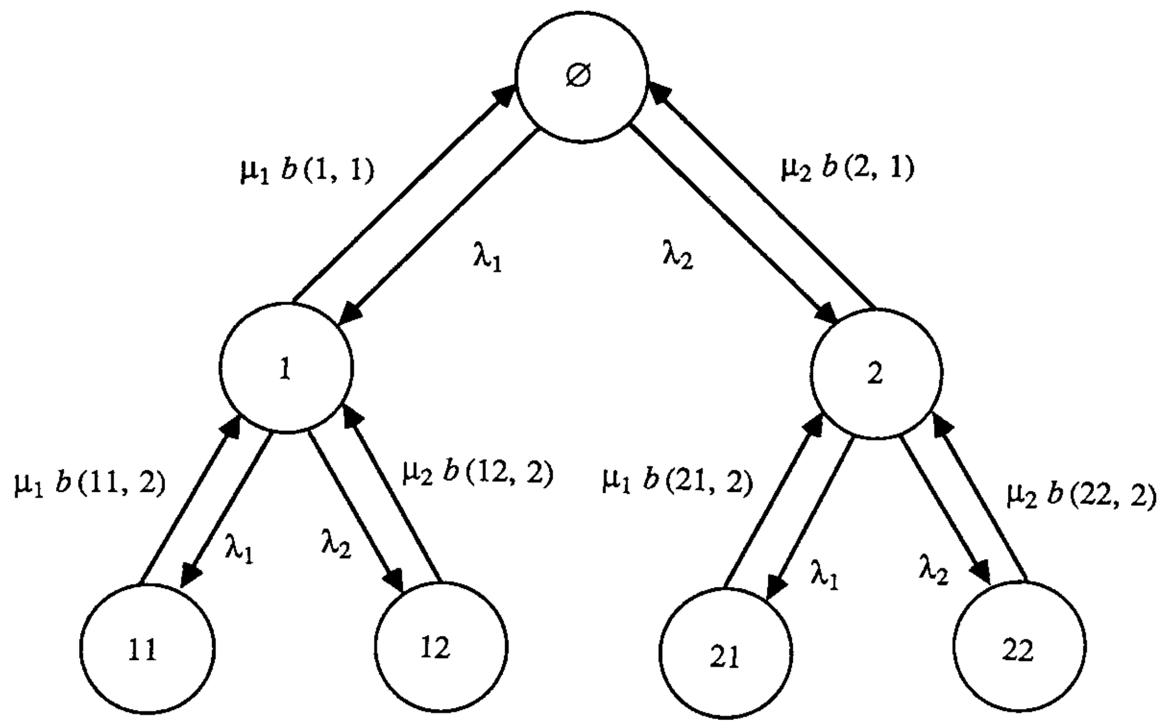


Figure 6. Natural state graph for the TOQA LCFSPR center with fully state dependent capacity function (for two classes).

First we must establish the following lemma. Recall that we have already defined $\kappa(X, r)$ to be the number of times that the symbol r occurs in the string X.

Lemma 1. Let $X_i = x_1 x_2 \cdots x_i$ and $Y_j = y_1 y_2 \cdots y_j$ where $x_i, y_j \in \Sigma$. Suppose that $\kappa(X_n, r) = \kappa(Y_n, r)$ for all $r = 1, 2, \ldots, R$, i.e. both X_n and Y_n contain the same number of customers of each class. If $f(\cdot, \cdot)$ is a function from pairs of nonnegative integers to nonnegative reals, then for each $r \in \Sigma$

$$\sum_{\substack{i=1 \\ x_i=r}}^{n} f[\kappa(X_n, r), \kappa(X_i, r)] = \sum_{\substack{j=1 \\ y_j=r}}^{n} f[\kappa(Y_n, r), \kappa(Y_i, r)] \tag{9}$$

Proof. We observe that the left hand side of Equation (9) is equal to

$$\sum_{k=1}^{\kappa(X_n, r)} f[\kappa(X_n, r), k] \tag{10}$$

and the right hand side is equal to

$$\sum_{l=1}^{\kappa(Y_n, r)} f[\kappa(Y_n, r), l] \tag{11}$$

Since $\kappa(X_n, r) = \kappa(Y_n, r)$, Expressions (10) and (11) are equal, whence follows Equation (9). ■

The next theorem generalizes the classical BCMP state dependent form given in Equation (1).

Proposition 1. Let $g_1(\cdot, \cdot), g_2(\cdot, \cdot), \ldots, g_R(\cdot, \cdot)$ and $h(\cdot)$ be functions mapping nonnegative integers to nonnegative reals. If the capacity function of a TOQA service center has the following form then the center has the M $\Rightarrow$ M property:

$$b(X_n, i) = g_{x_i}[\kappa(X_n, x_i), \kappa(X_i, x_i)]\, h(n) \tag{12}$$

where $X_i \equiv x_1 x_2 \cdots x_i$ for $i = 1, 2, \ldots, n$.

Proof. We begin by defining the macrostate

$$(n_1, n_2, \ldots, n_R) \equiv \vec{n} \equiv \{X \in \Sigma^* : \kappa(X, r) = n_r, r = 1, 2, \ldots, R\}$$

to contain all natural states with n_1 class 1 customers, n_2 class 2 customers, etc. If we let $\vec{e}_r$ represent the unit basis vector consisting of a 1 in the rth dimension and 0s everywhere else and $P_{\Delta t}(X \to Y)$ be the conditional probability of being in state Y within Δt time units, given that the system is now in state X, then we may express the probability of transitioning from macrostate $\vec{n}$ to $\vec{n} - \vec{e}_r$ as the conditional sum of the probabilities of transitioning from all microstates comprising $\vec{n}$ to microstates containing one less customer of class r:

$$P_{\Delta t}(\vec{n} \to \vec{n} - \vec{e}_r) = \sum_{X_n \in \vec{n}} \sum_{\substack{i=1 \\ x_i = r}}^{n} P_{\Delta t}(X_n \to x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n) \frac{P(X_n)}{P(\vec{n})} \tag{13}$$

We know that for the natural Markov process (14)

$$P_{\Delta t}(X_n \to x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n) = R(X_n \to x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n)\, \Delta t + o(\Delta t)$$

We substitute Equation (14) back into Equation (13). This yields

$$P_{\Delta t}(\vec{n} \to \vec{n} - \vec{e}_r) = \sum_{X_n \in \vec{n}} \sum_{\substack{i=1 \\ x_i = r}}^{n} [R(X_n \to x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n)\, \Delta t + o(\Delta t)] \frac{P(X_n)}{P(\vec{n})} \tag{15}$$

Dividing Equation (15) by Δt and taking the limit as Δt approaches 0 gives the rate of transition from $\vec{n}$ to $\vec{n} - \vec{e}_r$:

$$R(\vec{n} \to \vec{n} - \vec{e}_r) = \sum_{X_n \in \vec{n}} \sum_{\substack{i=1 \\ x_i = r}}^{n} R(X_n \to x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n) \frac{P(X_n)}{P(\vec{n})}$$

The quantity inside the summations is the transition rate due to a class r departure, conditioned on being in microstate X_n. We apply Equation (12) and further simplify Equation (15) to

$$\sum_{X_n \in \vec{n}} \sum_{\substack{i=1 \\ x_i = r}}^{n} b(X_n, i)\, \mu_r \frac{P(X_n)}{P(\vec{n})} = \sum_{X_n \in \vec{n}} \sum_{\substack{i=1 \\ x_i = r}}^{n} g_r[\kappa(X_n, r), \kappa(X_i, r)]\, h(n)\, \mu_r \frac{P(X_n)}{P(\vec{n})}$$

$$= h(n)\, \mu_r \sum_{X_n \in \vec{n}} \frac{P(X_n)}{P(\vec{n})} \sum_{\substack{i=1 \\ x_i = r}}^{n} g_r[\kappa(X_n, r), \kappa(X_i, r)]$$

By Lemma 1 the last summation

$$\sum_{\substack{i=1 \\ x_i = r}}^{n} g_r[\kappa(X_n, r), \kappa(X_i, r)] = \sum_{k=1}^{\kappa(X_n, r)} g_r[\kappa(X_n, r), k]$$

has identical values for all states $X_n \in \vec{n}$. We denote this value by $g_r(n_r)$ since it depends only on $\kappa(X_n, r) = n_r$. Thus

$$R(\vec{n} \to \vec{n} - \vec{e}_r) = g_r(n_r)\, h(n)\, \mu_r \frac{1}{P(\vec{n})} \sum_{X_n \in \vec{n}} P(X_n)$$

$$= g_r(n_r)\, h(n)\, \mu_r$$

since the summation evaluates to $P(\vec{n})$. For transitions that result from arrivals it is also easy to see that

$$R(\vec{n} \rightarrow \vec{n}+\vec{e}_r) = \lambda_r$$

The process $\Psi \equiv \{\vec{n}(t) : t \geq 0\}$ is a Markov process. One will notice from Equation (13) that the probability of being in a particular next state is influenced only by the current state and in no way depends on prior history. When the system comes into state $\vec{n}$ a state change occurs as the result of an arrival or a service completion. These arrival and service mechanisms clearly depend only on the current state of the system. It is as though, upon entering state $\vec{n}$, arrival and departure timers are set for each of the R customer classes, and the first timer to expire fires the appropriate transition.

We will show that Ψ is also reversible. From reversibility we will then derive the M $\Rightarrow$ M property. To show that Ψ is reversible we verify the following proposed solution for the service center:

$$P(\vec{n}) = P(\varnothing) \left[\prod_{r=1}^{R} \prod_{j=1}^{n_r} \frac{\lambda_r}{\mu_r\, g_r(j)} \right] \prod_{k=1}^{n} \frac{1}{h(k)} \tag{16}$$

With respect to the process Ψ a typical global balance equation for the service center is

$$P(\vec{n}) \sum_{r=1}^{R} \lambda_r + P(\vec{n}) \sum_{r=1}^{R} g_r(n_r)\, h(n)\, \mu_r$$

$$= \sum_{r=1}^{R} P(\vec{n}+\vec{e}_r)\, g_r(n_r+1)\, h(n+1)\, \mu_r + \sum_{r=1}^{R} P(\vec{n}-\vec{e}_r)\, \lambda_r$$

and Equation (16) clearly satisfies this set of equations. The only pairs of macrostates with non-trivial transition rates are pairs of the form $\vec{n}$ and $\vec{n}-\vec{e}_r$. It is straightforward to check that in Ψ we have

$$\lambda_r\, P(\vec{n}-\vec{e}_r) = g_r(n_r)\, h(n)\, \mu_r\, P(\vec{n}) \tag{17}$$

by substituting the expression for the equilibrium state probabilities provided in Equation (16). All communicating state pairs in Ψ are of the form $\vec{n}$ and $\vec{n}-\vec{e}_r$ — other pairs have trivial transition rates between them. Equation (17) holds for all feasible $\vec{n}$ and r, which is a necessary and sufficient condition for the reversibility of Ψ [Kel79, HeS82].

Since the process Ψ is reversible, the departure process of each class must be Poisson, as discussed in the proof of Theorem 1. Hence the center is a PFSC. ■

The capacity function

$$b(X_n, i) = g_{x_i}[\kappa(X_n, x_i), \kappa(X_i, x_i)]\, h(n)$$

may be rewritten as

$$b(X_n, i) = g_{x_i}(n_{x_i}, n_i)\, h(n)$$

where n_i designates the ith customer's position among all customers of its class. If we view the center as having a separate queue for each class then Proposition 1 states that the jth customer in queue r is served with rate $g_r(n_r, i)\, h(n)$. Thus the capacity allocated to this customer depends on the total number of customers in the center, the number of customers in its class, and the position it holds in its class queue. For example, a variant of processor sharing might allocate $u_{r,i}/n \times 100$ percent of the server to the ith customer of the rth queue rather than the customary $1/n \times 100$ percent. It is assumed that

$$\sum_{i=1}^{n_r} u_{r,i} = n_r$$

which ensures that the one server is completely shared by all n customers, though not necessarily with the same fairness policy of standard processor sharing.

4.3 Nonexponential PFSCs

Up to now we have only considered service centers with exponentially distributed service requirements. Next we examine LCFSPR centers with the exponentiality constraint relaxed.

Corollary 2 is somewhat surprising in that it demonstrates that LCFSPR service centers are extremely robust. One may choose any capacity function for an LCFSPR service center with exponential service requirements and the center will always be M $\Rightarrow$ M. Even more surprising, then, is the fact that Corollary 2 remains true even if the service requirement of each class has an *arbitrary* distribution.

We start by modifying the service center model presented in Section 3 to include Coxian (or branching Erlang) distributions [Cox55] for each class. A Coxian distribution has a rational Laplace transform and may be represented by a generalized method of stages with branching, as shown in Figure 7. Each class r will have parameters $\mu_{r,1}, \mu_{r,2}, \ldots, \mu_{r,K_r}$ corresponding to each of the K_r stages of exponential service. At stage k the class r customer proceeds to stage $k+1$ with probability $p_{r,k}$ and leaves service with probability $q_{r,k}$. We assume that $p_{r,K_r} = 0$ and $q_{r,K_r} = 1$. The Coxian distribution can approximate any distribution with a Laplace transform by fitting a rational function to the given transform. The rational function may then be inverted to yield the approximate distribution.

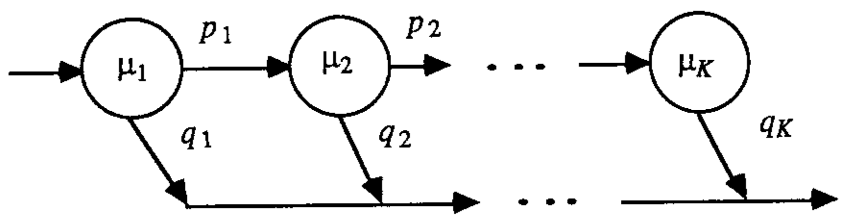


Figure 7. Coxian distribution. Each circle represents an exponential stage of service.

The natural state description may also be generalized. We take as the state of the queue $(x_1, k_1)(x_2, k_2) \cdots (x_n, k_n)$ where k_i is the stage of service that the ith customer is in and x_i is its class.

Lemma 2. If $p_i + q_i = 1$ for $1 \le i < K$ and $q_K = 1$, then

$$q_1 + q_2 p_1 + q_3 p_2 p_1 + \cdots + q_K p_{K-1} p_{K-2} \cdots p_1 = 1 \tag{18}$$

Proof. The proof is by induction on K. The basis case of $K = 1$ is obvious. Assume the proposition is true for $K - 1$. Let $p_1, p_2, \ldots, p_{K-1}$ and $q_1, q_2, \ldots, q_K$ be sequences satisfying the conditions of the lemma. Writing

$$q_1 + q_2 p_1 + q_3 p_2 p_1 + \cdots + q_{K-1} p_{K-2} p_{K-3} \cdots p_1 + q_K p_{K-1} p_{K-2} \cdots p_1 \tag{19}$$

we notice that the last two terms of the series reduce to

$$q_{K-1} p_{K-2} p_{K-3} \cdots p_1 + q_K p_{K-1} p_{K-2} \cdots p_1 = p_{K-2} p_{K-3} \cdots p_1$$

because $p_{K-1} + q_{K-1} = 1$ and $q_K = 1$. Now we may rewrite the expression of Equation (19) as

$$q_1 + q_2 p_1 + q_3 p_2 p_1 + \cdots + p_{K-2} p_{K-3} \cdots p_1$$

which we know is equal to 1 by the induction hypothesis. Thus Equation (19) evaluates to 1 and the lemma is proved. ■

Theorem 2. Assume that a (TOQA) service center uses the LCFSPR queueing discipline. Suppose that the Laplace transform of each class's service requirement is a rational function. Let the capacity function of the service center be arbitrarily dependent upon the extended natural state of the queue (including the customer's class and stage of service). The center is a PFSC with probability state distribution given by

$$P(y_1 y_2 \cdots y_n) = P(\varnothing) \prod_{i=1}^{n} \frac{\sigma_{y_i}}{b(y_1 y_2 \cdots y_i, i)\, \mu_{y_i}} \tag{20}$$

where we define $\sigma_{r,k} \equiv \lambda_r p_{r,1} p_{r,2} \cdots p_{r,k-1}$ for each class r and stage k.

Proof. We show that Equation (20) satisfies both the global balance equations and the M $\Rightarrow$ M condition. Let $y_i \equiv (x_i, k_i)$ for $i = 1, 2, \ldots, n$. Now define the following quantities:

$$A \equiv P(y_1 y_2 \cdots y_n) \sum_{r=1}^{R} \lambda_r$$

$$B \equiv P(y_1 y_2 \cdots y_n)\, b(y_1 y_2 \cdots y_n, n)\, \mu_{y_n}$$

$$C \equiv \sum_{r=1}^{R} \sum_{k=1}^{K_r} P[y_1 y_2 \cdots y_n(r, k)]\, b[y_1 y_2 \cdots y_n(r, k), n+1]\, q_{r,k}\, \mu_{r,k}$$

$$D \equiv P[y_1 y_2 \cdots y_{n-1}(x_n, k_n{-}1)]\, b[y_1 y_2 \cdots y_{n-1}(x_n, k_n{-}1), n]\, p_{x_n, k_n-1}\, \mu_{x_n, k_n-1}$$

$$E \equiv P(y_1 y_2 \cdots y_{n-1})\, \lambda_{x_n}$$

For $1 < k_n \le K_{x_n}$ the balance equations are

$$A + B = C + D$$

and for $k_n = 1$ the balance equations are

$$A + B = C + E$$

The relationships $B = D$ and $B = E$ may be verified by substituting the expression for $P(\cdot)$ of Equation (20) into B, D, and E. To see that $A = C$ we start with

$$\sum_{k=1}^{K_r} P[y_1 y_2 \cdots y_n(r, k)]\, b[y_1 y_2 \cdots y_n(r, k), n+1]\, q_{r,k}\, \mu_{r,k}$$

$$= \sum_{k=1}^{K_r} P(\varnothing) \left[\prod_{i=1}^{n} \frac{\sigma_{y_i}}{b(y_1 y_2 \cdots y_i, i)\, \mu_{y_i}} \right] \sigma_{r,k}\, q_{r,k}$$

$$= \left[P(\varnothing) \prod_{i=1}^{n} \frac{\sigma_{y_i}}{b(y_1 y_2 \cdots y_i, i)\, \mu_{y_i}} \right] \sum_{k=1}^{K_r} \sigma_{r,k}\, q_{r,k}$$

The summation of this last equation may be simplified to

$$\lambda_r\, [q_{r,1} + q_{r,2}\, p_{r,1} + \cdots + q_{r,K_r}\, p_{r,K_r-1}\, p_{r,K_r-2} \cdots p_{r,1}]$$

By Lemma 2 this reduces to λ_r. Thus $A = C$ and the balance equations hold: $A + B = C + D$ when $1 < k_n \le K_{x_n}$ and $A + B = C + E$ when $k_n = 1$. This establishes that Equation (20) is the expression for the equilibrium state probability distribution.

We may adapt Muntz's [Mun72, Mun83] characterization of an M $\Rightarrow$ M LCFSPR exponential service center to nonexponential service requirements:

$$\sum_{k=1}^{K_r} P[y_1 y_2 \cdots y_n(r, k)]\, b[y_1 y_2 \cdots y_n(r, k), n+1]\, q_{r,k}\, \mu_{r,k} = P(y_1 y_2 \cdots y_n)\, \lambda_r \qquad (21)$$

for each class $r \in \Sigma$. We substitute Equation (20) into the left hand side of Equation (21) to obtain

$$\sum_{k=1}^{K_r} P(\emptyset)\left[\prod_{i=1}^{n} \frac{\sigma_{y_i}}{b(y_1 y_2 \cdots y_i,\, i)\,\mu_{y_i}}\right] \frac{\sigma_{r,k}}{b[y_1 y_2 \cdots y_n(r,\, k),\, n+1]\,\mu_{r,k}}$$

$$\times\, b[y_1 y_2 \cdots y_n(r,\, k),\, n+1]\, q_{r,k}\, \mu_{r,k}$$

$$= \left[P(\emptyset) \prod_{i=1}^{n} \frac{\sigma_{y_i}}{b(y_1 y_2 \cdots y_i,\, i)\,\mu_{y_i}}\right] \sum_{k=1}^{K_r} q_{r,k}\, \sigma_{r,k}$$

$$= P(y_1 y_2 \cdots y_n)\, \lambda_r$$

The last equality, which is the right hand side of Equation (21), follows from Lemma 2. Hence the M $\Rightarrow$ M condition holds for each class r and the center is a PFSC. ■

Notice that the capacity function of Theorem 2 is even more general than that of Corollary 3. For the nonexponential case the capacity function depends not only on the arrangement of customers in the queue but also on the stage of service that each customer has achieved. This may be interpreted as saying that the capacity function is completely dependent upon the queue state, in the sense that the state of the queue may in general be specified by means of the technique of supplementary variables [Kle75]. The state is taken to be the list of customers in the queue and the amount of service each one has already received [CHT77].

4.4 TOQACI PFSCs

Returning our attention to service centers with exponential service requirements, we consider the use of the TOQACI state description. A TOQACI center may be viewed as temporally ordering customers and providing the capability of distinguishing individual customers from each other. Given a state description of such fine granularity, it is interesting to attempt a characterization of TOQACI PFSCs. We focus on reversible TOQACI centers. The following proposition formalizes the notion that if one kept strict accounting of customer identities in a queueing system then, for the center to appear the same when time is reversed, a customer would have to depart from the same queue position to which it arrived, viz. the tail of the queue.

Proposition 2. The only reversible TOQACI service center is the LCFSPR center.

Proof. It was seen in Corollary 2 that any TOQA LCFSPR center is reversible. Adding sequence numbers for customer identification does not affect the reversibility. The state graph clearly retains its tree shape. Now suppose that a TOQACI center is reversible. The only state changes that result from arrivals are of the form $\langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_n, m_n\rangle \rightarrow \langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_n, m_n\rangle\langle r, m_n{+}1\rangle$. It is clear that for reversibility to hold the only allowable state changes resulting from departures must be of the form $\langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_n, m_n\rangle$

$\rightarrow \langle x_1, m_1\rangle\langle x_2, m_2\rangle \cdots \langle x_{n-1}, m_{n-1}\rangle$. These transitions are due to the departure of the customer at the tail of the queue. Since they are the only kinds of departure transitions allowed the center is LCFSPR. ■

5 Applications

The results of the previous section, while of theoretical interest, also have relevance to practical modeling activities. Some of these application areas are discussed below.

5.1 LCFSPR PFSCs

LCFSPR models do not predominate in the study of computer systems. Of the four types of BCMP service centers, LCFSPR is probably the type least used in practice. Nonetheless, this discipline is used in actual systems, e.g. for processor scheduling in some interactive systems [SaC81] and for approximating systems with high dispatch frequency of priority tasks [LZG84].

LCFSPR may also be used to approximate heavily loaded carrier-sense-multiple-access with collision detection (CSMA/CD) channels, such as the Ethernet or IEEE 802.3 media access protocol for local networks. As delays on heavily loaded CSMA/CD channels can be significant, it is important to be able to incorporate the impact of these delays on jobs that execute across the network. The binary exponential backoff algorithm used for contention resolution increases the likelihood that a packet that has just arrived will sense the channel idle and initiate transmission, whereas previously arrived packets that were involved in collisions will have rescheduled transmission for a later time. Thus the youngest packet on the channel can often sneak through before older packets that have been queued for a while. This produces a last-come-first-served effect. Certainly the service rate for a recently arrived packet is affected by the channel state, i.e. by all packets queued and waiting for their backoff timers to reset. These effects could be empirically captured and abstractly represented by the capacity function for a LCFSPR center.

The heavily loaded Ethernet model is shown in Figure 8. The N stations are all connected to the channel which is represented by the LCFSPR center. Customers are processed at a station and then routed to the next station after experiencing a channel access delay at the LCFSPR center. Such a cycle may repeat until the customer leaves the network.

Distributed real time computing systems often dedicate processors to the servicing of externally generated asynchronous events. An event arrives to the processor and immediately generates an interrupt that causes the current context to be saved and pushed onto the stack. The preempting event is then processed and eventually dispatched to an appropriate processor, whereupon the preempted, suspended task is resumed. LCFSPR scheduling is necessary to minimize interrupt latency and may in some situations actually reduce context switching over-

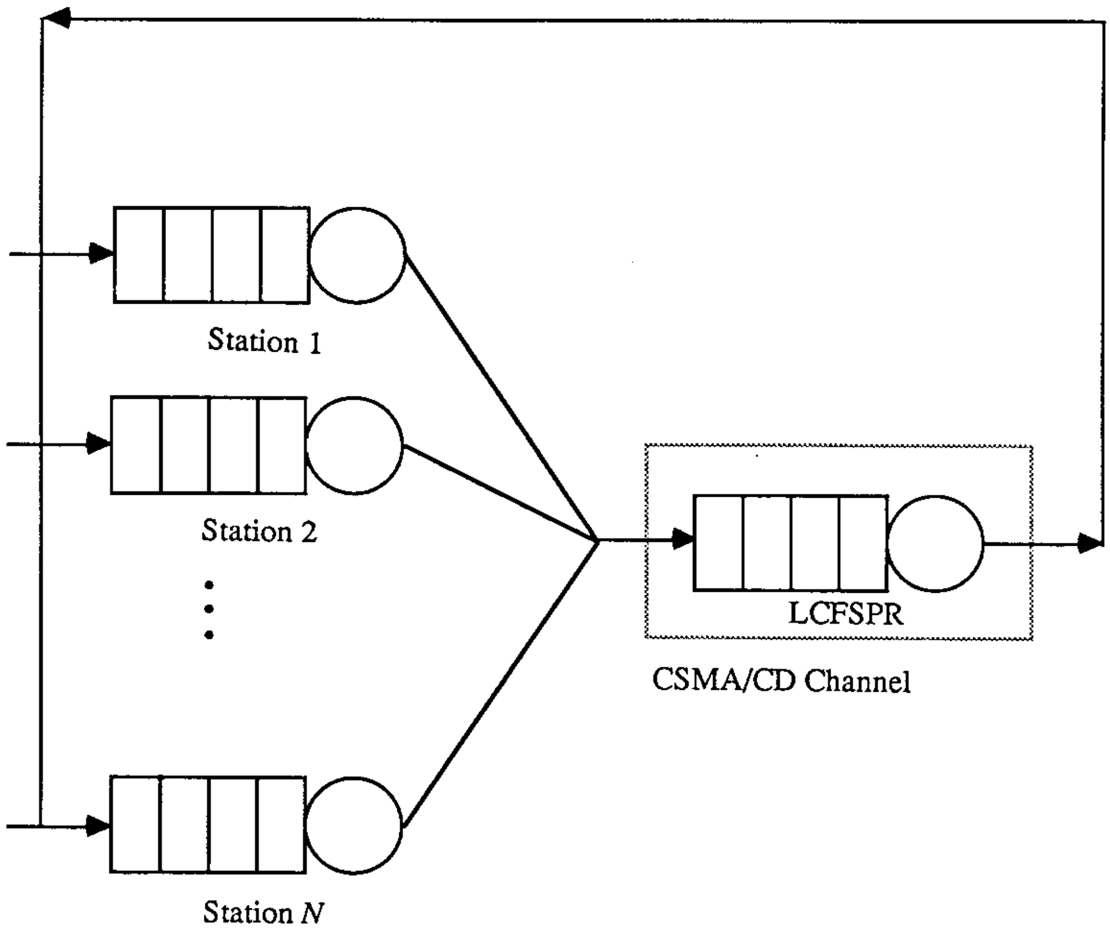


Figure 8. Modeling a heavily loaded CSMA/CD channel as an LCFSPR state dependent service center. There are N stations connected to a common channel.

head. Modeling this kind of device is a straightforward application of the LCFSPR service center.

Each of these systems has peculiarities that give rise to state sensitive service rates. Corollary 3 and its generalization, Theorem 3, provide guarantees that we can model such systems as PFSCs, regardless of their service requirement distributions or state dependencies. The power of Theorem 3 leads to the informal observation that any real queueing system using the LCFSPR discipline can be modeled as a PFSC and thus easily incorporated in a product form queueing model. Of course it is necessary to assume that the system's arrival and service statistics obey the usual independence conditions. This generality of service characteristics (capacity and distribution) makes LCFSPR PFSCs unique among all known service center types.

5.2 *HMS PFSCs*

Corollary 3 broadens our repertoire of tools for analyzing queueing networks. The HMS service center introduced there provides a useful technique for modeling heterogeneous computer systems. This extends the modeling capabilities of tractable queueing networks into new domains.

In a distributed computing system based on the client-server paradigm, a particular job may remotely invoke specific functions at specially designated server machines [Spe82]. It is often desirable to support a service at several different machines. When a function is invoked the client machine locates an available server by broadcasting a service request. Any server machine that is available may respond with a service reply indicating to the client that it is immediately ready to process the client's function. The client will choose the first responding server and dispatch its function to it. Allowing the client to request the service and discover an available server before dispatch, rather than always sending the request to a particular server, means that the service may be provided by a pool of servers, which results in better availability, network transparency, and device independent software. The client may also receive service sooner if it delays dispatching its function invocation until it knows there is a free server. This is similar to the motivation for using a common queue in a bank rather than individual queues at each teller's window. The delays and processing required by the service requests and replies are negligible compared to the function processing time.

It is problematic to model this type of system with standard BCMP service centers. There seems to be no easy way to model the global queue that has access to a number of different (heterogeneous) servers. Nor is it obvious how to cause a customer to be routed to a server only if the server is not busy. The HMS center, however, models this type of system very accurately. The server pool corresponds to the K servers of the HMS center. The queue of the HMS center is assumed to be located at the client machine. The choice of which server the client will use is generally nondeterministic. Variable delays in processing the service request, formulating the service reply, and accessing the communications media tend to randomize the selection of where to dispatch the function. Thus it is quite acceptable to assume that a customer who arrives to an HMS center is randomly assigned to a free server.

The queueing network model of client-server interaction is shown in Figure 9. Jobs arrive to the client from an external source (which may be other service centers of a larger network). They are processed by the client until remote processing by a server is needed (note that there may be many server *pools*), whereupon the job is dispatched to the HMS, processed, and returned to the client. This cycle continues until the job completes and exits the client center. The probability of requiring remote service after completion of a job step is $p_{r,\text{remote}}$ for each job class r. The completion probability is simply its complement $p_{r,\text{local}}$.

6 Conclusion

This paper has examined several aspects of the role of state dependence in specifying service centers. By appropriately choosing a state dependent capacity function, new queueing disciplines may be constructed. It has been shown that there are structural properties of a center's Markov state graph that yield easily analyzed PFSCs. The four-cycle and path properties are powerful tools for discovering new and useful PFSCs. Two new PFSCs, the totally state dependent, nonexponential LCFSPR service center, and the HMS service center, have been identified by means of the techniques of Theorem 1 and its corollaries.

The significance of the path and four-cycle properties is that they may be used to characterize PFSCs without appeal to the center's state probability distribution. Other criteria for product form, e.g. the M $\Rightarrow$ M condition [Mun72, Mun83], local and station balance [CHT77], all employ the center's state probabilities in their formulations. Even the balanced property of [ChM83] is defined in terms of a nonintuitive characteristic function that must be derived or known *a priori*. Solving for the state probabilities of a system or class of systems is often a formidable task. The path and four-cycle properties may be verified without knowing a center's state probabilities. All that is required is to *check that simple arithmetic relationships hold among certain transition rates*. Since the properties are closely related to reversibility, it is clear that they are less general than other PFSC characterizations. For instance, Muntz [Mun72] points out that reversibility is stronger than the M $\Rightarrow$ M property. Nevertheless, the path and four-cycle properties hold for many known systems, including all of the BCMP PFSCs,[8] even those with state dependent service rates given by Equation (1). This is evidence of the wide applicability of the technique. It has also been used to discover new PFSCs. Even though the results were primarily presented in terms of the natural Markov process, it may be noted that they apply to queueing systems with different state descriptions, as in Corollary 2 and Theorem 4.

[8]Providing they have exponential service requirements.

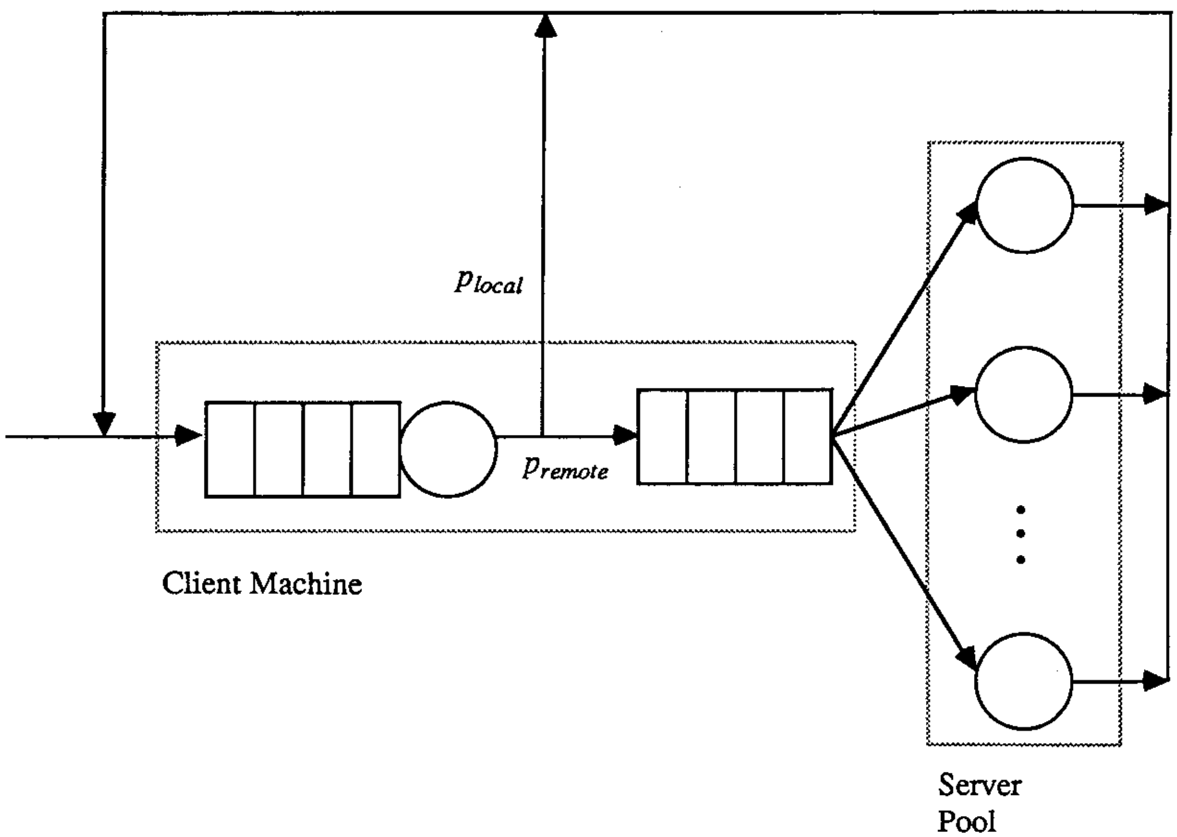


Figure 9. Model of a client-server system using an HMS service center.

Other aspects of state dependent servers may be profitably investigated. In particular, it may be worthwhile to examine the use of the techniques of this paper for constructing FESCs used in decomposition methods for the approximate analysis of queueing networks that violate product form. State dependent LCFSPR centers, HMS centers, and composite queues satisfying the hypotheses of Proposition 2 are candidate FESCs for modeling aggregate subnetworks. The appropriateness of such FESCs should be studied in the context of approximate analysis of queueing networks.

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