Graphical Models for Causal Inference

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Introduction

Why do we need graphs?

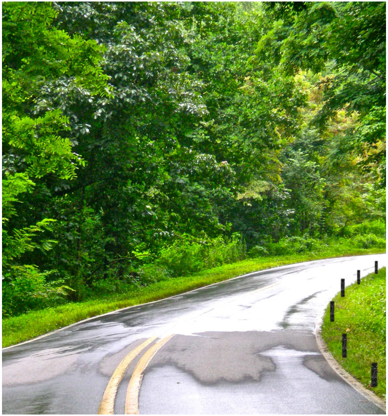

Figure: Motivating Example

Introduction

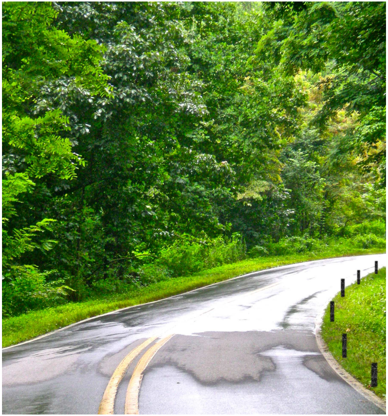

Figure: Motivating Example

Variables in the study:

- Season
- Sprinkler
- Rain
- Wetness of pavement(Wet)
- Slipperiness of pavement(Slippery)

Introduction

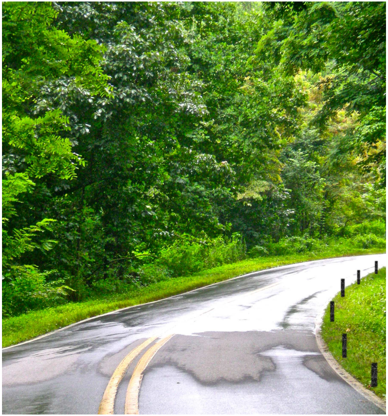

Figure: Motivating Example

# Variables	Table size
5	32
6	64
7	128
8	256
9	512
10	1,024
20	1,048,576
30	1,073,741,824

Introduction

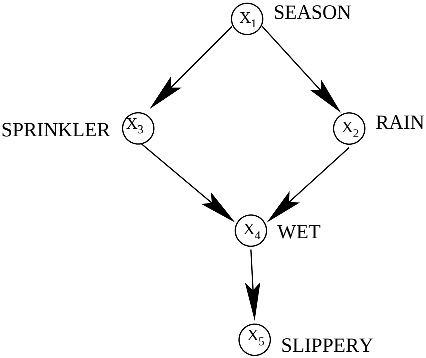

Figure: DAG Representation

Conditional Probability Distributions

- $P(X_1)\,:\,2$
- $P(X_3|X_1)\,:\,4$
- $P(X_2|X_1)\,:\,4$
- $P(X_4|X_2,X_3)\,:\,8$
- $P(X_5|X_4)\,:\,4$

Total # of Table Entries = 22

Graphs: Notations

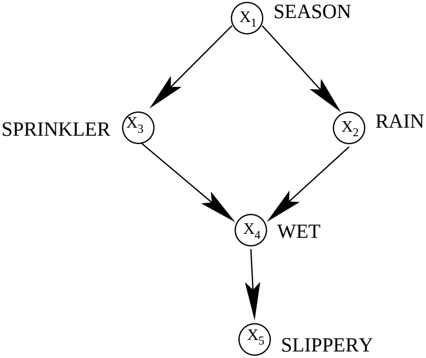

Figure: Bayesian Network representing dependencies

- Adjacent Nodes
- Root and Leaf Nodes
- Skeleton
- Path
- Kinship Terminology

Graphs: Notations

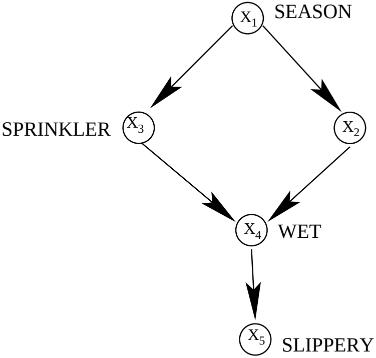

Figure: Bayesian Network representing dependencies

Chain $X_1 \rightarrow X_3 \rightarrow X_4 \rightarrow X_5$

$X_1 \rightarrow X_3 \rightarrow X_4 \rightarrow X_5$

Fork $X_3 \leftarrow X_1 \rightarrow X_2$

Collider $X_3 \rightarrow X_4 \leftarrow X_2$

Background Factors & Bi-directed Edges

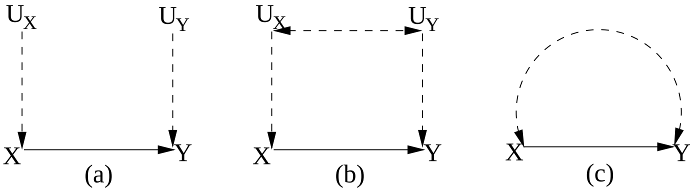

Figure: (a) Causal Model with background factors (b) & (c) Causal Model with correlated background factors

Note: Figure (a) expresses the assumption: $U_x \perp\!\!\!\perp U_y$ and Figure (b)& (c) express the assumption $U_x \not\perp\!\!\!\perp U_y$

Decomposing joint distribution-$P(V)$

How would you decompose joint distribution $P(V)$ into smaller distributions?

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By applying Chain rule

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Let $X_1, X_2, ..., X_n$ be any arbitrary ordering of nodes in a DAG.
$P(x_1, x_2, ..., x_n) = \prod_j P(x_j|x_1, ..., x_{j-1})$

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Is it possible that conditional probability of some variable X_j is not sensitive to all its predecessors?

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Is it possible that conditional probability of some variable X_j is not sensitive to all its predecessors?

Yes!

Markovian Parents

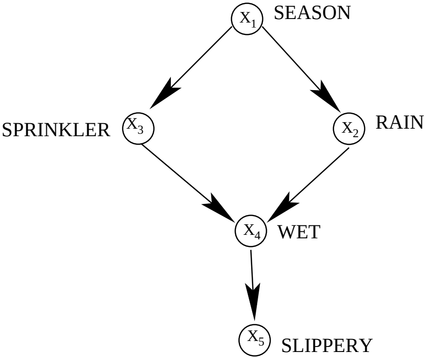
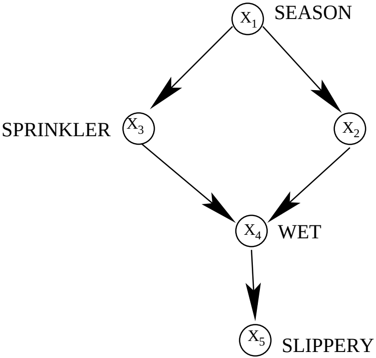


Figure: Bayesian Network representing dependencies

Markovian Parents

$X_1 : \phi$
$X_2 : \{X_1\}$
$X_3 : \{X_1\}$
$X_4 : \{X_2, X_3\}$
$X_5 : \{X_4\}$

Markovian Parents



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$X_1 : \phi$
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$X_4 : \{X_2, X_3\}$
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$$P(x_1, x_2, x_3, x_4, x_5) = P(x_1)P(x_2|x_1)P(x_3|x_1)P(x_4|x_2, x_3)P(x_5|x_4)$$

Markov Compatibility

Let $V = \{x_1, x_2, ..., x_n\}$ be the set of observed nodes and pa_i be the Markovian parents of x_i. Then,
$P(v) = P(x_1, x_2, .., x_n) = \prod_i P(x_i|pa_i)$.

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Definition (Markov Compatibility)

If a probability distribution P admits Markovian factorization of observed nodes relative to DAG G, we say that G and P are Markov compatible.

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Definition (Markov Compatibility)

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Example

X	Y	$P(X,Y)$
1	1	0.225
1	0	0.375
0	1	0.125
0	0	0.275

Markov Compatible DAGs:
$X \rightarrow Y$
$X \leftarrow Y$

Testing Markov Compatibility

Given a DAG G and distribution P, how can you conclude that P and G are compatible?

Testing Markov Compatibility

Given a DAG G and distribution P, how can you conclude that P and G are compatible?

- Parents shielding tests
  - non-descendants
  - predecessors
- d-separation

d-Separation

Definition

Let X, Y and Z be disjoint sets in DAG G. X and Y are d-separated by Z (written $(X \perp\!\!\!\perp Y|Z)_G$) if and only if Z blocks every path from a node in X to a node in Y.

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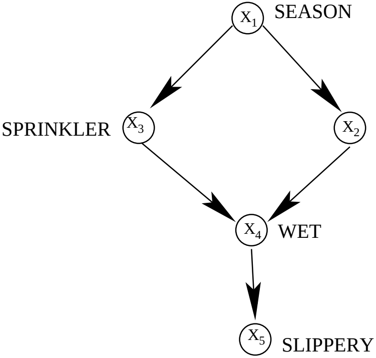
A path p is said to be d-separated(or blocked) by a set of nodes Z if and only if :
(1) p contains a chain $i \rightarrow m \rightarrow j$ or a fork $i \leftarrow m \rightarrow j$ such that the middle node m is in Z, or
(2) ...

Example: d-Separation



- $X_1 \amalg X_4 | X_2, X_3$
- $X_3 \amalg X_5 | X_4$
- $X_1 \amalg X_5 | X_4$

d-Separation

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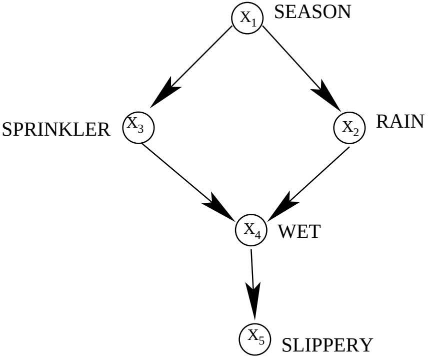
A path p is said to be d-separated(or blocked) by a set of nodes Z if and only if :

(1) p contains a chain $i \rightarrow m \rightarrow j$ or a fork $i \leftarrow m \rightarrow j$ such that the middle node m is in Z, or

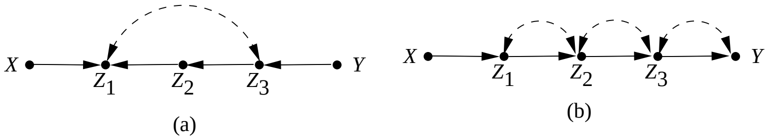
(2) p contains an inverted fork (or collider) $i \rightarrow m \leftarrow$ such that the middle node m is not in Z and such that no descendant of m is in Z.

Example: d-Separation



- $X_3 \amalg X_2 | X_1$

Example: d-Separation



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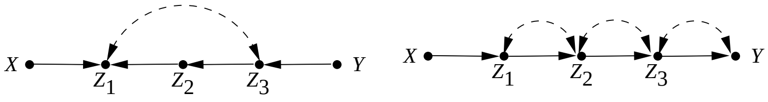
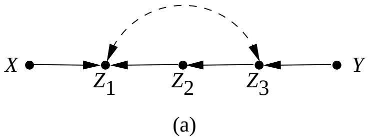
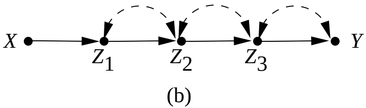


(a) (b)

Case (a): $X \amalg Y | \phi$

Example: d-Separation





Case (a): $X \amalg Y | \phi$

Case (b): $X \not\amalg Y$

d-Separation

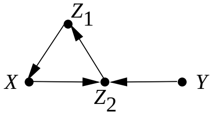
When is it impossible to d-separate 2 non-adjacent nodes X and Y?

Do we need to test all sets for possible separation?

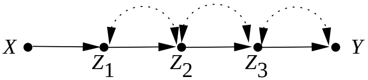
Inducing path

Definition

Path between 2 nodes X and Y is termed inducing if every non-terminal node on the path:
(i) is a collider and
(ii) an ancestor of either X or Y (or both)



(a)



(b)

Note: There are no separators for X and Y.

The Five Necessary Steps of Causal Analysis

Define Express the target quantity Q as property of the model M.

Assume Express causal assumptions in structural or graphical form.

Identify Determine if Q is identifiable.

Estimate Estimate Q if it is identifiable; approximate it, if it is not.

Test If M has testable implications

A “Mini” Turing Test in Causal Conversation

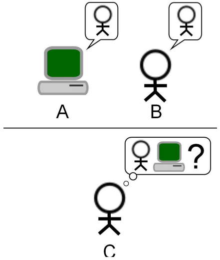


Figure: Turing Test

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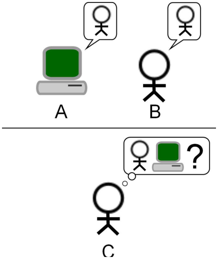

Figure: Turing Test

Input: Story

Question: What if? What is? Why?

Answers: I believe that...

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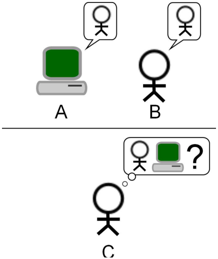
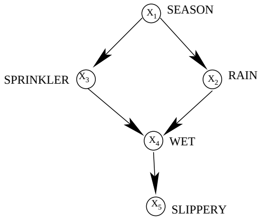


Figure: Turing Test

The Story



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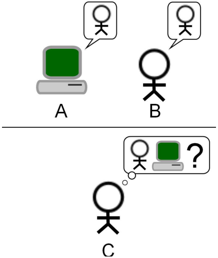

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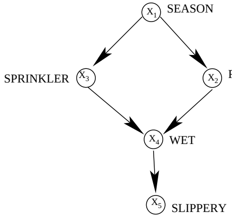

Q1: If the season is dry and the pavement is slippery, did it rain?

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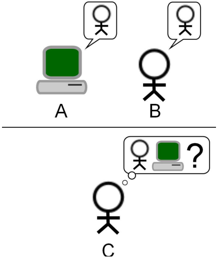

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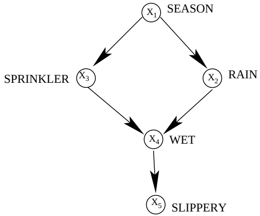
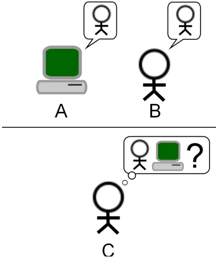
The Story



Q1: If the season is dry and the pavement is slippery, did it rain?
A1: Unlikely, it is more likely that the sprinkler was ON with a very slight possibility that it is not even wet.

A "Mini" Turing Test in Causal Conversation



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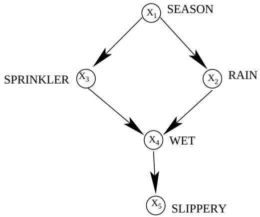
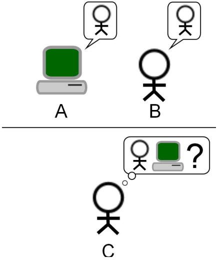


Q2: But what if we see that the sprinkler is OFF?

A “Mini” Turing Test in Causal Conversation



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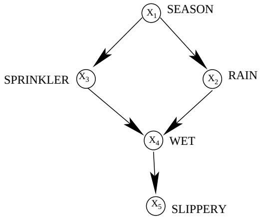

Q2: But what if we see that the sprinkler is OFF?
A2: Then it is more likely that it rained.
Without graphs,# of Table Entries = 32

A "Mini" Turing Test in Causal Conversation

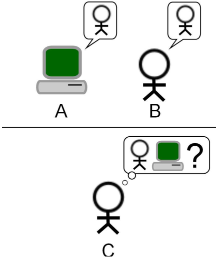

Figure: Turing Test

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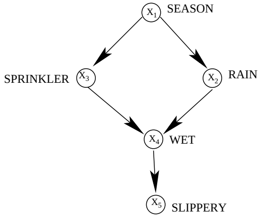


Q3: Do you mean that if we actually turn the sprinkler ON, the rain will be less likely?

A "Mini" Turing Test in Causal Conversation

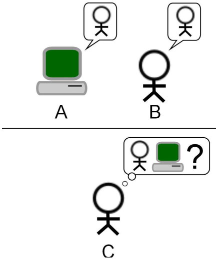
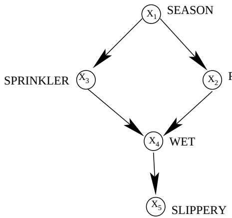


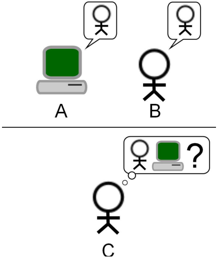
Figure: Turing Test

The Story



Q3: Do you mean that if we actually turn the sprinkler ON, the rain will be less likely?
A3: No, the likelihood of rain would remain the same but the pavement would surely get wet.
Without graphs,# of Table Entries = 32 * 32

A “Mini” Turing Test in Causal Conversation



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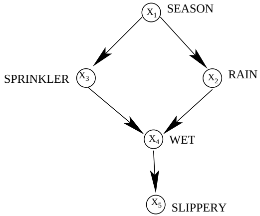


Q4: Suppose we see that the sprinkler is ON and pavement is wet. What if the sprinkler were OFF?

A "Mini" Turing Test in Causal Conversation

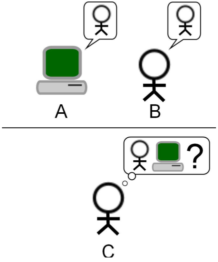
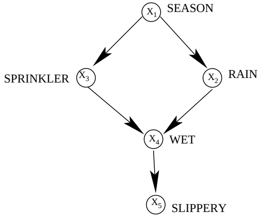


Figure: Turing Test

The Story



Q4: Suppose we see that the sprinkler is ON and pavement is wet. What if the sprinkler were OFF?
A4: The pavement would be dry because the season is likely to be dry
Without graphs, what would be the # of table entries?

Interventions

Query: Would the pavement be slippery if we *make sure that the* the sprinkler is on?

Compute: $P(x_5|do(x_3))$

May be equivalently represented as:

(a) $P(x_5|\hat{x_3})$

(b) $P_{x_3}(x_5)$

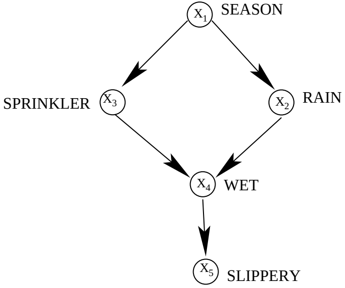


Figure: DAG before intervention

Interventions

Compute: $P(x_5|do(x_3))$

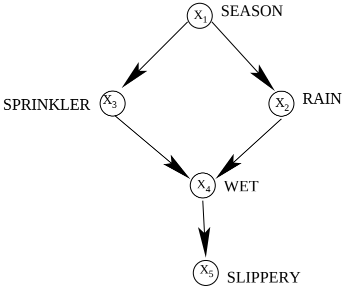


Figure: DAG before intervention

$$P(v) = P(x_1)P(x_2|x_1)P(x_3|x_1)P(x_4|x_2, x_3)P(x_5|x_4)$$

Interventions

Compute: $P(x_5|do(x_3))$

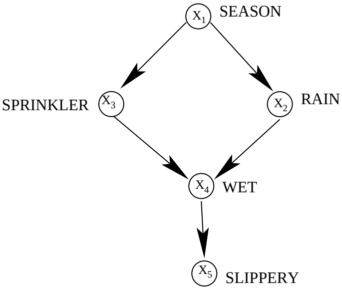


Figure: DAG before intervention

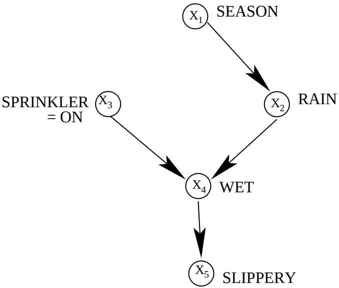
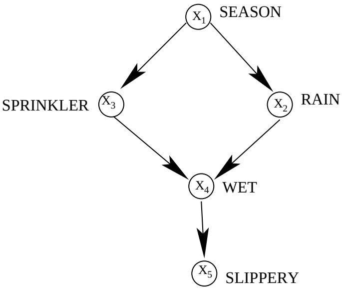


Figure: DAG after intervention

$$P(v) = P(x_1)P(x_2|x_1)\,\underline{P(x_3|x_1)}\,P(x_4|x_2,x_3)P(x_5|x_4)$$
$$P(x_1,x_2,x_4,x_5|do(x_3)) = P(x_1)P(x_2|x_1)P(x_4|x_2,x_3)P(x_5|x_4)$$

Interventions

Compute: $P(x_5|do(x_3))$



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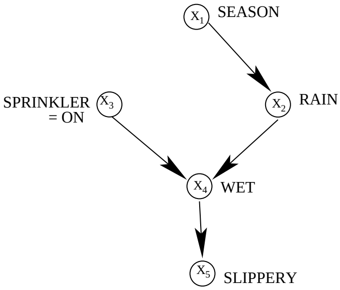


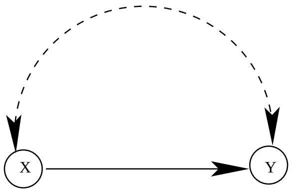
Figure: DAG after intervention

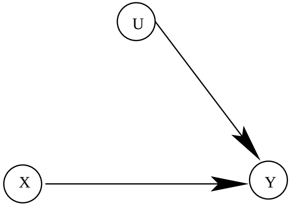
$$P(v) = P(x_1)P(x_2|x_1)\ \underline{P(x_3|x_1)}\ P(x_4|x_2,x_3)P(x_5|x_4)$$

$$P(x_5|do(x_3)) = \sum_{x_1,x_2,x_4} P(x_1)P(x_2|x_1)P(x_4|x_2,x_3)P(x_5|x_4)$$

Note: $P(x_5|do(x_3)) \neq P(x_5|x_3)$ i.e. Doing $\neq$ Seeing

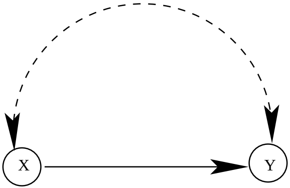
Examples

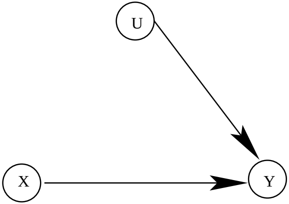




Question : Can you estimate $P(y|do(x))$, given $P(x, y)$?

Examples





Question : Can you estimate $P(y|do(x))$, given $P(x,y)$?

NO!

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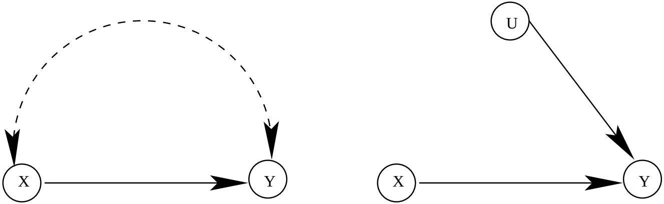
Examples



Question : Can you estimate $P(y|do(x))$, given $P(x, y)$?

NO!

$$P(x, y) = \sum_u P(x, y, u) = \sum_u P(y|x, u)P(x|u)P(u)$$

$$P(y|do(x)) = \sum_u P(y|x, u)P(u)$$

Identifiability

Definition

Let $Q(M)$ be any computable quantity of a model M.

Identifiability

Definition

Let $Q(M)$ be any computable quantity of a model M. We say that Q is identifiable in a class M of models if, for any pairs of models M_1 and M_2 from M, $Q(M_1) = Q(M_2)$ whenever $P_{M_1}(v) = P_{M_2}(v)$.

Estimating causal effect

Adjustment for direct causes

Compute: $P(y|\hat{x})$

$P(x, y, z, w) = P(y|x, w)P(x|z)P(w|z)P(z)$

$P(y, z, w|do(x)) = P(y|x, w)P(w|z)P(z)\ \frac{P(x|z)}{P(x|z)}$

$P(y, z, w|do(x)) = \frac{P(x,y,z,w)}{P(x|z)}$

$P(y|do(x)) = \sum_{z,w} P(yw|x, z)P(z) = \sum_{z} P(y|x, z)P(z)$

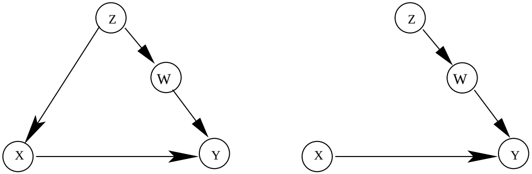


Figure: DAGs before and after intervention

Theorem (Adjustment for direct causes)

Let PA_i denote the set of direct causes of X_i and let Y be any set of variables disjoint of $\{X_i \cup Pa_i\}$. The causal effect of X_i on Y is given by:

$$P(y|\hat{x}_i) = \sum_{pa_i} P(y|x_i, pa_i)P(pa_i)$$

where $P(y|x_i, pa_i)$ and $P(pa_i)$ represent pre-interventional probabilities.

Example: Adjustment for direct causes

Query: Would the pavement be slippery if we *make sure that the* the sprinkler is on?

$$P(x_5|\hat{x_3}) = \sum_{x_1} P(x_5|x_3, x_1)P(x_1)$$

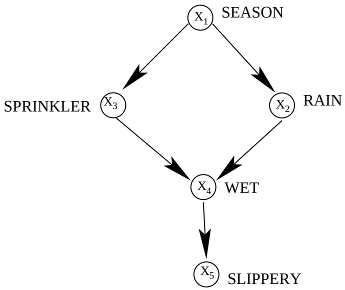


Figure: DAG before intervention

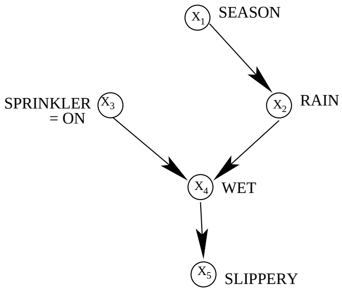
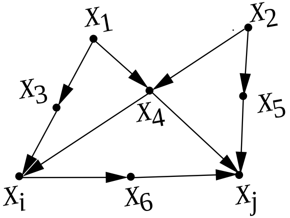


Figure: DAG after intervention

Estimating Causal Effect

Compute: $P(X_j|do(X_i))$
How can we find a set Z of concomitants that are sufficient for identifying causal effect?



Back-door Criterion for Identifiability

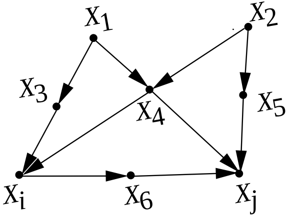
Definition (Pearl-1993)

A set of variables Z satisfies the back-door criterion relative to an ordered pair of variables (X_i, X_j) in a DAG G if:

(i) no node in Z is a descendant of X_i; and

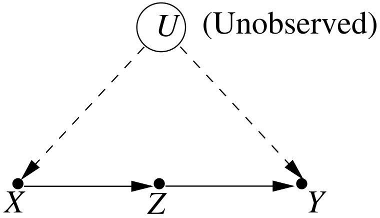
(ii) Z blocks every path between X_i and X_j that contains an arrow into X_i.

$P(x_j|do(x_i)) = \sum_z P(x_j|x_i, z)P(z)$



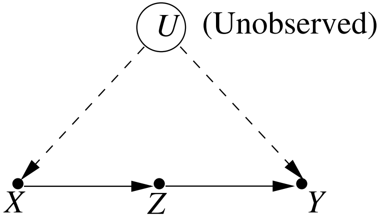
Estimating causal effect: $P(y|do(x))$

- Can you adjust for direct cause?



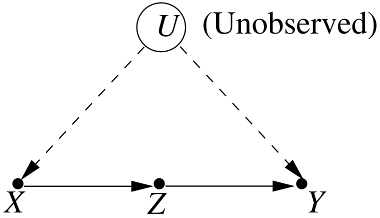
Estimating causal effect: $P(y|do(x))$

- Can you adjust for direct cause? NO!



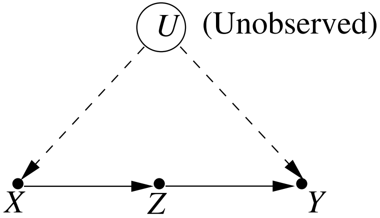
Estimating causal effect: $P(y|do(x))$

- Can you apply backdoor criterion?



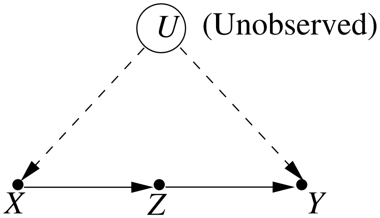
Estimating causal effect: $P(y|do(x))$

- Can you apply backdoor criterion? NO!



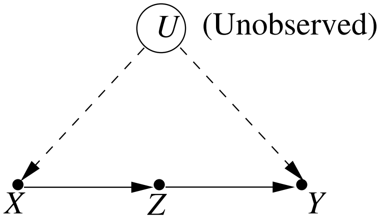
Estimating causal effect: $P(y|do(x))$

- Is $P(y|do(x))$ identifiable?



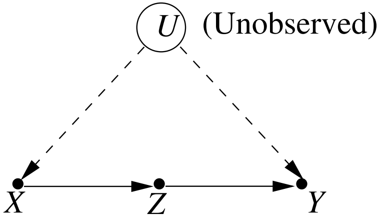
Estimating causal effect: $P(y|do(x))$

- Is $P(y|do(x))$ identifiable? YES!



Estimating causal effect: $P(y|do(x))$

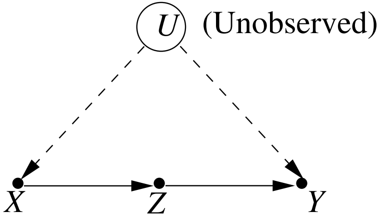
Given: $P(y|\hat{x}) = \sum_z P(y|\hat{z})P(z|\hat{x})$



Estimating causal effect: $P(y|do(x))$

Given: $P(y|\hat{x}) = \sum_z P(y|\hat{z})P(z|\hat{x})$
$P(z|\hat{x}) = P(z|x)$
$P(y|\hat{z}) = \sum_{x'} P(y|x', z)P(x')$

Therefore,
$P(y|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x')$



Front-door Criterion for Identifiability

Definition (Pearl-1995)

A set of variables Z satisfies the front-door criterion relative to an ordered pair of variables (X_i, X_j) in a DAG G if:

(i) Z intercepts all directed paths from X to Y; and
(ii) there is no unblocked back-door path from X to Z; and
(iii) all back-door paths from Z to Y are blocked by X

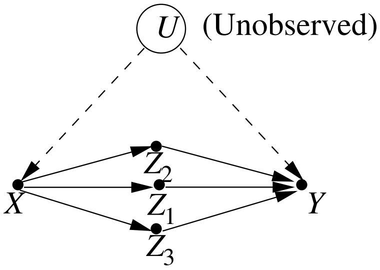


Figure: Frontdoor criterion is satisfied by $Z = \{Z_1, Z_2, Z_3\}$

Front-door Adjustment

If Z satisfies the front door criterion relative to (X, Y) and if $P(x, z) > 0$, then the causal effect of X on Y is identifiable and is given by:

$$P(y|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z) P(x')$$

Estimating causal effect $P(y|do(x))$

How can you syntactically derive claims about interventions?

Estimating causal effect $P(y|do(x))$

How can you syntactically derive claims about interventions?

- do-calculus

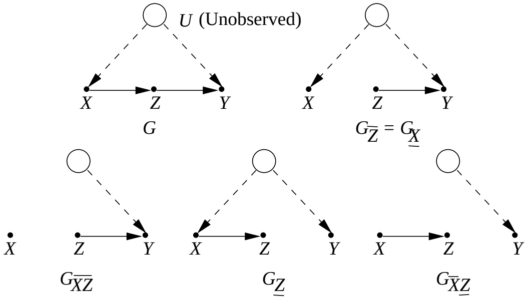
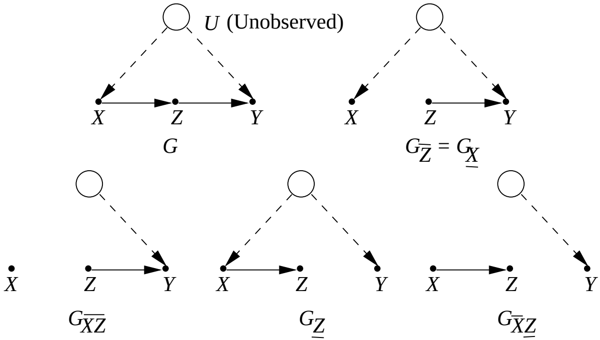


Figure: Subgraphs of G used in the derivation of causal effects.

do-Calculus-[Pearl-1995]

Rule-1 Insertion or deletion of observations

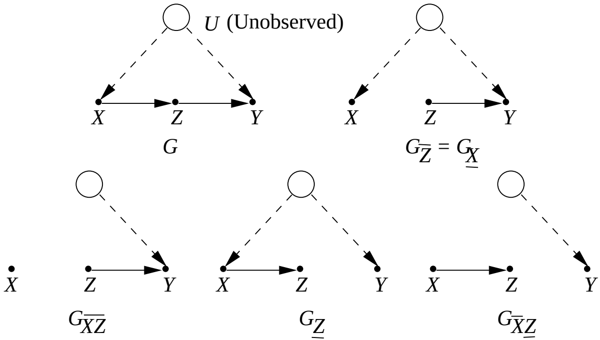
$P(y|\hat{x}, z, w) = P(y|\hat{x}, w)$ if $(Y \amalg Z|X, W)_{G_{\overline{X}}}$



do-Calculus-[Pearl-1995]

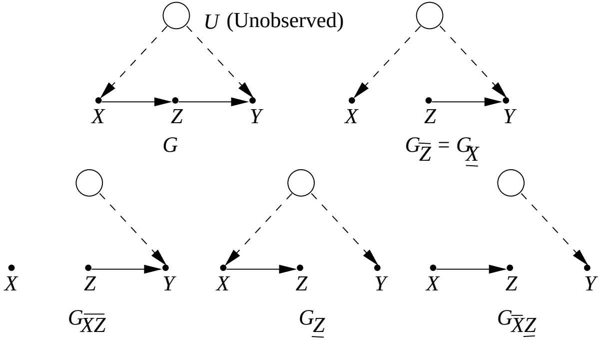
Rule-2 Action/Observation exchange

$P(y|\hat{x},\hat{z},w) = P(y|\hat{x},z,w)$ if $(Y \amalg Z|X,W)_{G_{\overline{X}\underline{Z}}}$

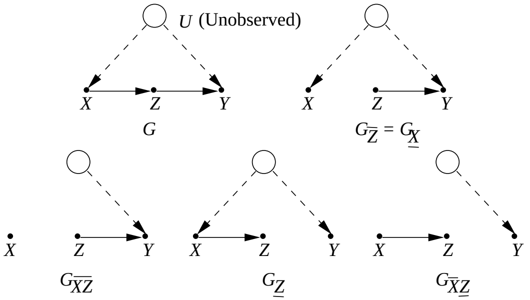


do-Calculus-[Pearl-1995]

Rule-3 Insertion or deletion of actions
$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w)$ if $(Y \amalg Z|X, W)_{G_{\overline{X}, \overline{Z(W)}}}$
where $Z(W)$ is the set of Z nodes that are not ancestors of any W node in $G_{\overline{X}}$



Deriving causal effect using do-calculus



Compute: $P(y|\hat{z})$

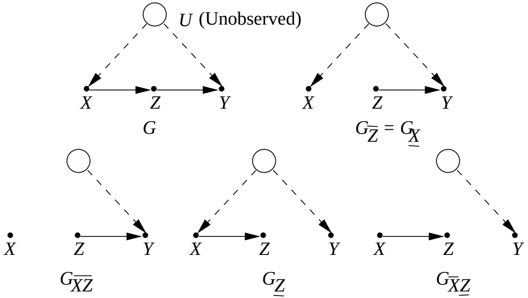
$P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$

$P(x|\hat{z}) = P(x)$ since $(Z \perp\!\!\!\perp X)_{G_{\overline{Z}}}$

$P(y|x, \hat{z}) = P(y|x, z)$ since $(Z \perp\!\!\!\perp Y|X)_{G_{\underline{Z}}}$

$P(y|\hat{z}) = \sum_x P(y|x, z)P(x)$

Prove: $P(y|\hat{x}) = \sum_z P(y|\hat{z})P(z|\hat{x})$



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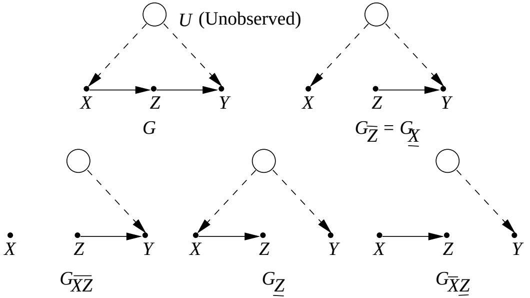
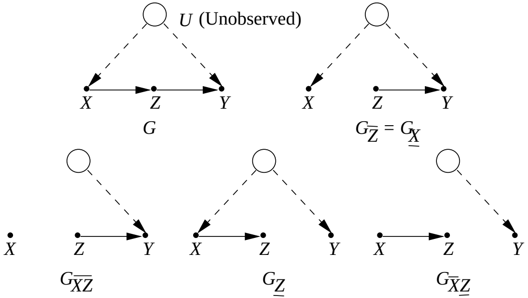
Prove: $P(y|\hat{x}) = \sum_z P(y|\hat{z})P(z|\hat{x})$



$$P(y|\hat{x}) = \sum_z P(yz|\hat{x}) = \sum_z P(y|\hat{x}z)P(z|\hat{x})$$

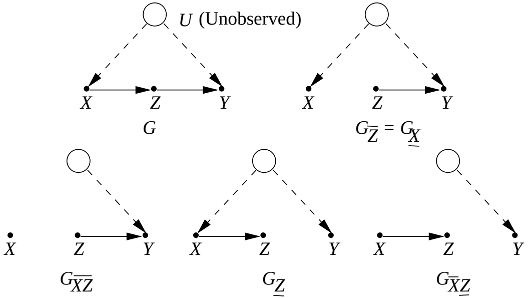
Prove: $P(y|\hat{x}) = \sum_z P(y|\hat{z})P(z|\hat{x})$



$$P(y|\hat{x}) = \sum_z P(yz|\hat{x}) = \sum_z P(y|\hat{x}z)P(z|\hat{x})$$

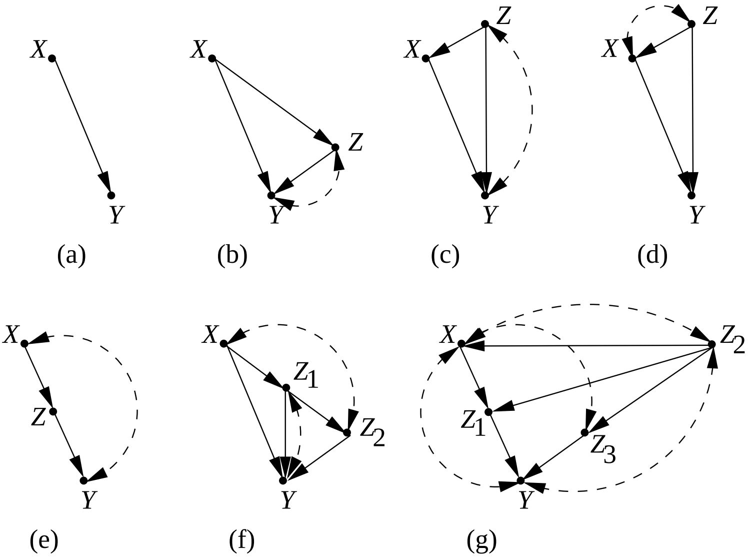
$P(y|z\hat{x}) = P(y|\hat{z}\hat{x})$ since $Y \perp\!\!\!\perp Z$ in $G_{\overline{X}\underline{Z}}$

Prove: $P(y|\hat{x}) = \sum_z P(y|\hat{z})P(z|\hat{x})$

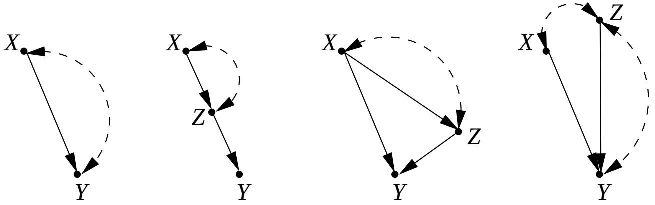


$$P(y|\hat{x}) = \sum_z P(yz|\hat{x}) = \sum_z P(y|\hat{x}z)P(z|\hat{x})$$
$$P(y|z\hat{x}) = P(y|\hat{z}\hat{x}) \text{ since } Y \perp\!\!\!\perp Z \text{ in } G_{\overline{X}\underline{Z}}$$
$$= P(y|\hat{z}) \text{ since } Y \perp\!\!\!\perp X \text{ in } G_{\overline{ZX}}$$

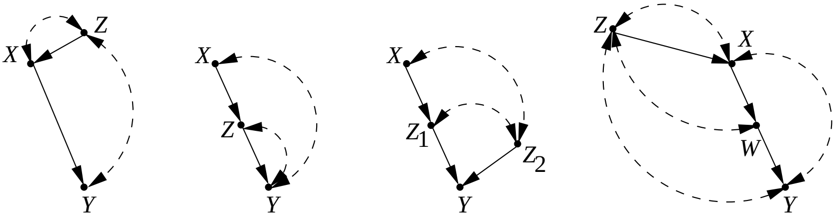
Graphical Models in which $P(y|\hat{x})$ is Identifiable



Graphical Models in which $P(y|\hat{x})$ is not Identifiable



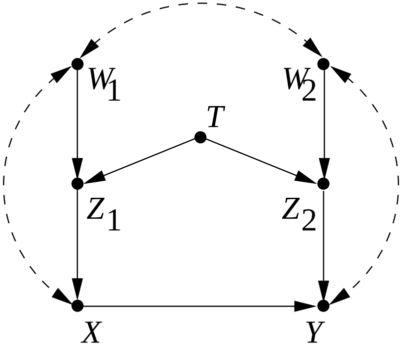
(a) (b) (c) (d)



(e) (f) (g) (h)

C-components and C-factor

Two variables are said to be in the same C-component if they are connected by a path comprising of only bi-directional edges[Tian & Pearl, 2002].



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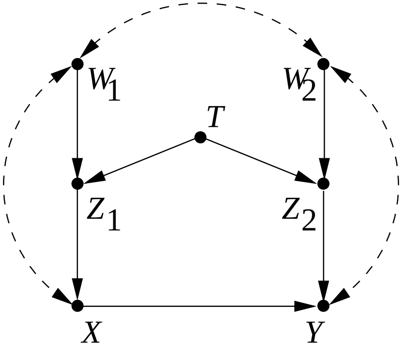
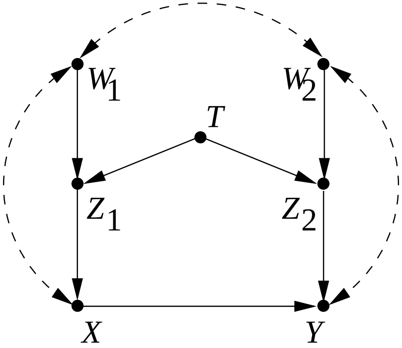


$S_1 = \{X, Y, W_1, W_2\}$
$S_2 = \{Z_1\}$
$S_3 = \{Z_2\}$
$S_4 = \{T\}$

C-components and C-factor

Two variables are said to be in the same C-component if they are connected by a path comprising of only bi-directional edges[Tian & Pearl, 2002].



$S_1 = \{X, Y, W_1, W_2\}$
$S_2 = \{Z_1\}$
$S_3 = \{Z_2\}$
$S_4 = \{T\}$

C-factor: $Q[S_i](v) = P_{v \setminus s_i}(s_i)$

Identifiability of C-factor

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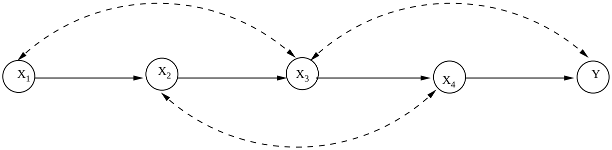
Lemma (Tian & Pearl, 2002)

Let a topological order over V be $V_1 < V_2 < ... < V_n$ and let $V^{(i)} = \{V_1, V_2, ..., V_i\}$, $i = 1, ..., n$ and $V^{(0)} = \phi$. For any set C, let G_C denote the subgraph of G composed only of variables in C. Then:

(i) Each C-factor $Q_j, j = 1, ..., k$ is identifiable and is given by:

$$Q_j = \prod_{\{i:V_i \in S_j\}} P(v_i|v^{(i-1)})$$

Example: Identifiability of C-factor



Admissible order: $X_1 < X_2 < X_3 < X_4 < Y$

$Q_1 = P(x_4|x_1, x_2, x_3)P(x_2|x_1)$

$Q_2 = P(y|x_1, x_2, x_3, x_4)P(x_3|x_1, x_2)P(x_1)$

Necessary and Sufficient condition for identifiability of $P_x(v)$

Theorem (Tian & Pearl, 2002)

Let X be a singleton. $P_x(v)$ is identifiable if and only if there is no bi-directed path connecting X to any of its children.

Necessary and Sufficient condition for identifiability of $P_x(v)$

Theorem (Tian & Pearl, 2002)

Let X be a singleton. $P_x(v)$ is identifiable if and only if there is no bi-directed path connecting X to any of its children. When $P_x(v)$ is identifiable, it is given by:

$$P_x(v) = \frac{P(v)}{Q^X} \sum_x Q^X,$$

where Q^X is the c-factor corresponding to the c-component S^X that contains X.

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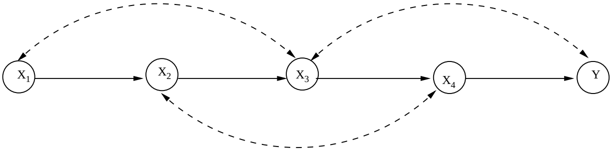
Example: Necessary and Sufficient condition for identifiability of $P_x(v)$



Admissible order: $X_1 < X_2 < X_3 < X_4 < Y$

$Q_1 = P(x_4|x_1, x_2, x_3)P(x_2|x_1)$

$Q_2 = P(y|x_1, x_2, x_3, x_4)P(x_3|x_1, x_2)P(x_1)$

$$P_{x_1}(x_2, x_3, x_4, y) = Q1 \sum_{x_1} Q2$$
$$= P(x_4|x_1, x_2, x_3)P(x_2|x_1)$$
$$\sum_{x_1} P(y|x_1, x_2, x_3, x_4)P(x_3|x_1, x_2)P(x_1)$$

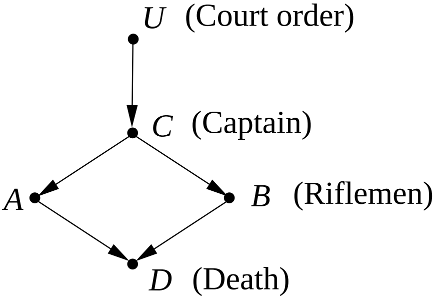
Causal Effect Identifiability

Identification of $P_x(y|z)$ where $X \cap Y \cap Z = \phi$ and X is not necessarily a singleton, [Shpitser & Pearl,2006]

- Hedge Criterion
- IDC - Sound and Complete Algorithm

Counterfactuals

Query: Would the prisoner be dead had rifleman A not shot him, given that the prisoner is dead and rifleman A shot him?



Counterfactuals

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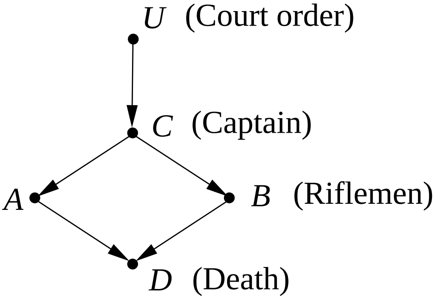
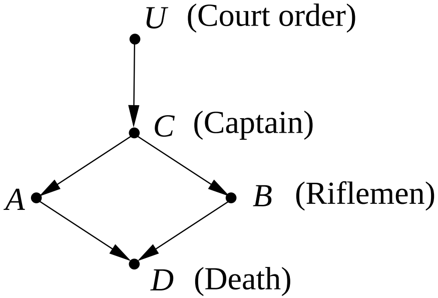


- Abduction

Counterfactuals

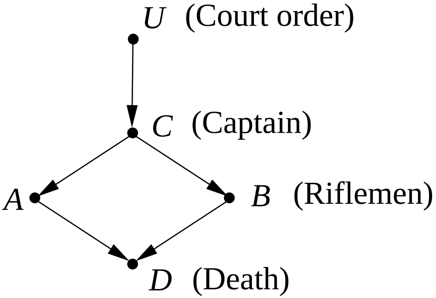
Query: Would the prisoner be dead had rifleman A not shot him, given that the prisoner is dead and rifleman A shot him?



- Abduction
- Intervention

Counterfactuals

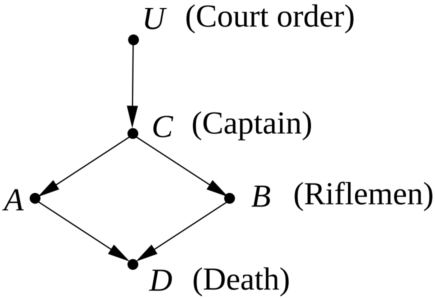
Query: Would the prisoner be dead had rifleman A not shot him, given that the prisoner is dead and rifleman A shot him?



- Abduction
- Intervention
- Prediction

Counterfactuals

Query: Would the prisoner be dead had rifleman A not shot him, given that the prisoner is dead and rifleman A shot him?



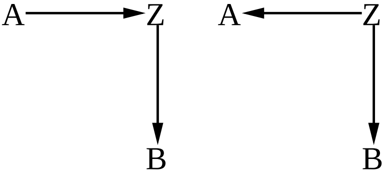
Model M	**Model** $M_{\neg A}$
$C = U$	$C = U$
$A = C$	$\neg A$
$B = C$	$B = C$
$D = A \vee B$	$D = A \vee B$
Facts: D	**Facts:** U
Conclusions:	**Conclusions:**
U, A, B, C, D	$U, \neg A, B, C, D$

Markov Equivalence

Given 2 models, is there a test that would tell them apart?

Definition

Two graphs G_1 and G_2 are said to be Markov equivalent if every d-separation condition in one also holds in the other. .



(G_1) (G_2)

Markov Equivalence

Given 2 models, is there a test that would tell them apart?

Definition

Two graphs G_1 and G_2 are said to be Markov equivalent if every d-separation condition in one also holds in the other. .

Are these DAGs Markov Equivalent?

Hard to enumerate all separation conditions.

Observational Equivalence

Theorem (Verma & Pearl 1990)

Two DAGs are observationally equivalent iff they have the same sets of edges and the same sets of v-structures, that is, two converging arrows whose tails are not connected by an arrow.

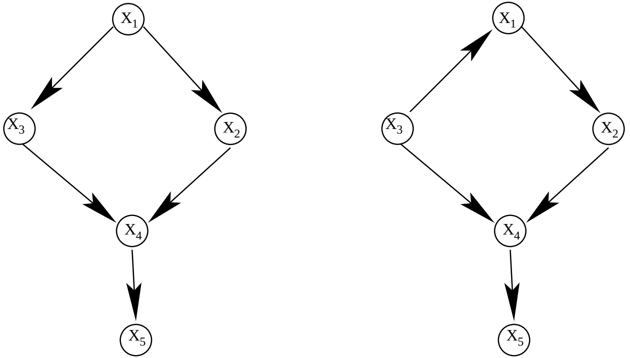


Figure: Observationally Equivalent DAGs

Markov Equivalence and Observational Equivalence

If two DAGs are Markov Equivalent, then they are Observationally Equivalent as well. True/False?

Markov Equivalence and Observational Equivalence

If two DAGs are Markov Equivalent, then they are Observationally Equivalent as well. True/False?
True if all variables are observed(i.e. no bi-directed edges) and False otherwise.

Markov Equivalence and Observational Equivalence

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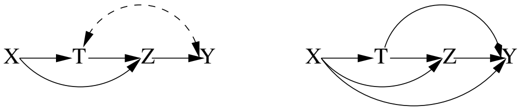


Figure: DAGs that are Markov Equivalent but not Observationally Equivalent

How would you distinguish between the two?

- Verma Constraints (Refer slide:115)

Ancestral Graphs

Definition (Ancestral Graphs)

A graph which may contain directed or bi-directed edges is ancestral if:
(i) there are no directed cycles
(ii) whenever there is an edge $X \longleftrightarrow Y$, then there is no directed path from X to Y or from Y to X.

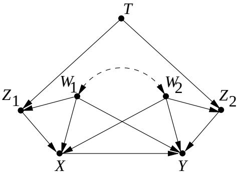


Figure: Ancestral graph

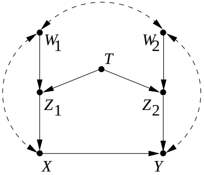


Figure: Not an Ancestral graph

Maximal Ancestral Graphs (MAGs)

Definition (Spirtes & Richardson, 2002)

An ancestral graph is said to be maximal if, for every pair of non-adjacent nodes X, Y there exists a set Z such that X and Y are d-separated conditional on Z.

Figure: DAG and its corresponding MAG

Construction of a MAG

$Given$: DAG G
Step-1: Construct a graph M comprising of:
(i) all nodes in G
(ii) all uni-directional edges in G

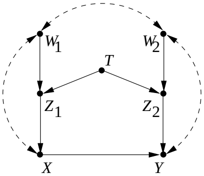


Figure: DAG G

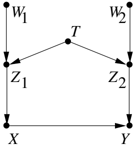


Figure: Graph M

UCLA

Construction of a MAG

Step-2: For every bi-directed edge $A \leftrightarrow B$ in G,
(i) add $A \rightarrow B$ to M if A is an ancestor of B in G
(ii) add $A \leftarrow B$ to M if B is an ancestor of A in G
(iii) copy $A \leftrightarrow B$ to M if (i) and (ii) do not hold true

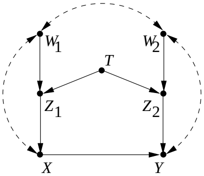


Figure: DAG G

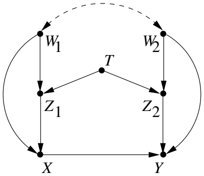


Figure: Graph M

Construction of a MAG

Step-3: For every pair of non-adjacent nodes A and B in G, connected by an *inducing path* ,
(i) add $A \rightarrow B$ to M if A is an ancestor of B in G
(ii) add $A \leftarrow B$ to M if B is an ancestor of A in G
(iii) add $A \leftrightarrow B$ to M if (i) and (ii) do not hold true

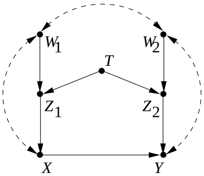


Figure: DAG G

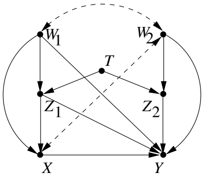


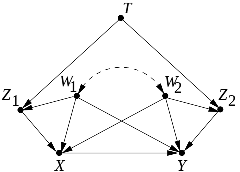
Figure: MAG M

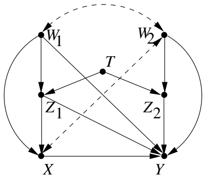
Markov Equivalence

Theorem

Two graphs G_1 and G_2 are said to be Markov equivalent if their MAGs are Markov Equivalent

Are these MAGs Markov Equivalent?





Note: Markov Equivalence in MAGs are easier to check

Complete criterion for determining Markov Equivalence of 2 MAGs: [Ali, Richardson and Spirtes, 2009]

Reversing an edge in a MAG

Definition (Screened Edge)

[Tian,2005] An edge $X \to Y$ is a screened edge in a MAG if $Pa(Y) = Pa(X) \cup \{X\}$ and $Sp(Y) = Sp(X)$[1].

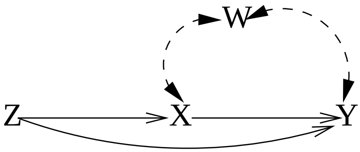


Figure: MAG with Screened Edge:$X \to Y$

[1] Nodes X and Y are spouses, if they are connected by a bi-directed edge.

Reversing an edge in a MAG

Theorem (Tian,2005)

Let M be a MAG with edge $X \rightarrow Y$ and M' be a graph with edge $X \longleftarrow Y$, otherwise identical to M. Then M' is a MAG that is Markov Equivalent to M if and only if $X \rightarrow Y$ is a screened edge in M.

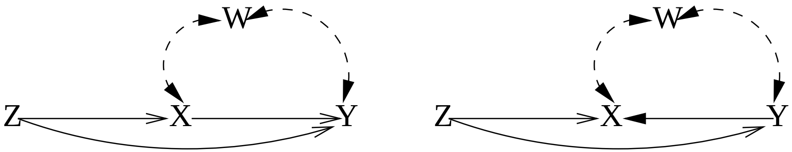


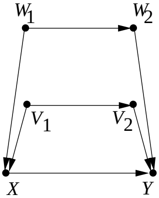
Figure: Markov Equivalent MAGs

Confounding Equivalence

Definition (Pearl and Paz,2009)

Define two sets, T and Z as c-equivalent (relative to X and Y), written $T \approx Z$, if the following equality holds for every x and y:

$$\sum_t P(y|x,t)P(t) = \sum_z P(y|x,z)P(z) \quad \forall x, y$$



Examples:

- $T = \{W_1, V_2\} \approx Z = \{W_2, V_1\}$
- $T = \{W_1, V_1\} \approx Z = \{W_2, V_2\}$
- $T = \{W_1, W_2\} \approx Z = \{W_1\}$
- $T = \{W_1, W_2\} \not\approx Z = \{W_2\}$

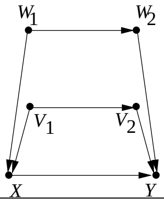
Note: C-equivalence is testable

Necessary and Sufficient Condition for C-Equivalence

Theorem (Pearl and Paz, 2009)

Let Z and T be two sets of variables containing no descendant of X. A necessary and sufficient condition for Z and T to be c-equivalent is that at least one of the following conditions hold:

- $X \perp\!\!\!\perp (Z \cup T) \mid (Z \cap T)$ *or*
- *Z and T are G-admissible*[2]



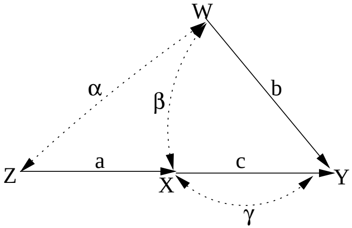
Examples:

- $T = \{W_1, V_2\} \approx Z = \{W_2, V_1\}$
- $T = \{W_1, V_1\} \approx Z = \{W_2, V_2\}$
- $T = \{W_1, W_2\} \approx Z = \{W_1\}$
- $T = \{W_1, W_2\} \not\approx Z = \{W_2\}$

[2] satisfies back-door criterion

Linear Models and Causal Diagrams

Assume all variables are normalized to have zero mean and unit variance.



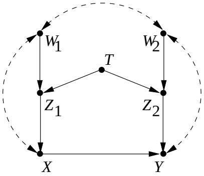
$$
\begin{aligned}
Z &= e_1 \\
W &= e_2 \\
X &= aZ + e_3 \\
Y &= bW + cX + e_4 \\
Cov(e_1, e_2) &= \alpha \neq 0 \\
Cov(e_2, e_3) &= \beta \neq 0 \\
Cov(e_3, e_4) &= \gamma \neq 0
\end{aligned}
$$

Which parameters can be identified?

Vanishing Regression Coefficient

Definition

For any linear model for a causal diagram D that may include cycles and bi-directed arcs, the partial correlation $\rho_{XY.Z}$ must vanish if and only if node X is d-separated from node Y by the variables of Z in D [Spirtes et al., 1997b].



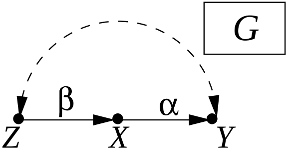
- $r_{TX.W_1Z_1} = 0$

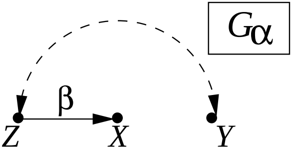
Find more

Single Door Criterion for Direct Effects

Theorem

Let G be any path diagram in which α is the path coefficient associated with link $X \to Y$ and let G_α denote the diagram that results when $X \to Y$ is deleted from G. The coefficient α is identifiable if there exists a set of variables Z such that :
(i) Z contains no descendant of Y and
(ii) Z d-separates X from Y in G_α
Moreover, if Z satisfies these two conditions, then α is equal to the regression coefficient $r_{YX.Z}$.





Instrumental Variables (IV)

Definition

A variable Z is an instrument relative to a cause X and an effect Y if:

- Z is independent of all error terms that have an influence on Y when X is held constant, and
- Z is **not** independent of X.

In linear systems, Causal effect of X on $Y = \frac{r_{ZY}}{r_{ZX}}$

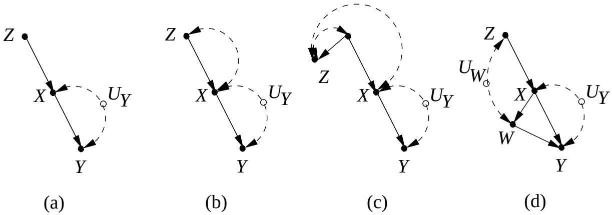


Figure: Z is an instrument in (a), (b) and (c) but not in (d)

Conditional Instrumental Variable

Definition (Brito & Pearl, 2002)

Z is an instrumental variable if $\exists$ a set W such that:

- W contains only non-descendants of Y
- W d-separates Z from Y in the sub-graph G_α obtained by removing the edge $X \rightarrow Y$
- W does not d-separate Z from X in G_α

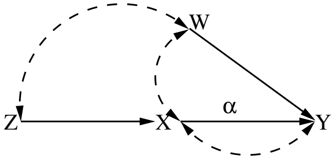
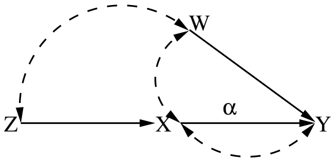


Figure: Graph G and corresponding subgraph G_α

Conditional Instrumental Variable

- Z is a conditional instrumental variable. Hence, $\alpha =$ Causal effect of X on $Y = \frac{r_{ZY.W}}{r_{ZX.W}}$
- W does not satisfy single-door criterion. So, α cannot be identified using single-door.



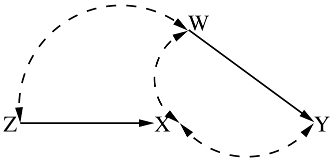
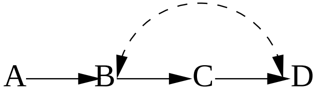
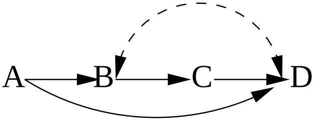


Figure: Graph G and corresponding subgraph G_α

Verma Constraints([Tian and Pearl, 2002])



(a)



(b)

$Q[\{B, D\}] =$
$\sum_u P(b|a, u)P(d|c, u)P(u)$
$P_{v\backslash d}(d) = \sum_u P(d|c, u)P(u)$
Also,
$Q[\{B, D\}] = P(d|a, b, c)P(b|a)$
$P_{v\backslash d}(d) = \sum_b P(d|a, b, c)P(b|a)$
$\sum_b P(d|a, b, c)P(b|a)$ is independent of a.

$Q[\{B, D\}] =$
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$P_{v\backslash d}(d) = \sum_u P(d|a, u, c)P(u)$
Also,
$Q[\{B, D\}] = P(d|a, b, c)P(b|a)$
$P_{v\backslash d}(d) = \sum_b P(d|a, b, c)P(b|a)$
$\sum_b P(d|a, b, c)P(b|a)$ is not independent of a.

Conclusions

Graphs are indispensable for:

- encoding causal knowledge
- identifying parameters and causal effects
- identifying testable implications

Go ahead and Exploit the Power of Graphs!

Thank You!