8 Appendix

8.1 Proof of Theorem 1

We first prove three lemmas.

Lemma 5. The z-specific PNS $P(y_x, y'_z | z)$ are bounded as follows:

$$\max \left\{ \frac{P(y_x | z) - P(y'_z | z)}{P(y_z | z) - P(y'_z | z)} \right\} \leq z\text{-PNS}$$

$$\min \left\{ \frac{P(y_x | z)}{P(y_z | z) + P(y'_z | z)} + \frac{P(y_z | z) - P(y'_z | z) + \frac{P(y_z | z)}{P(y'_z | z) + P(y'_z | z)}}{-\frac{P(y_z | z) - P(y'_z | z)}{P(y'_z | z) + P(y'_z | z)}} \right\} \geq z\text{-PNS}$$

Proof. Since for any three events $A$, $B$ and $C$, we have

$$P(A, B | C) \geq \max[0, P(A | C) + P(B | C) - 1]$$

therefore, we have

$z\text{-PNS} \geq \max[0, P(y_x | z) + P(y'_z | z) - 1]$

$$= \max[0, P(y_x | z) - P(y'_z | z)]$$

Also,

$$z\text{-PNS} = P(y_x, y'_z, x | z) + P(y_x, y'_z, x' | z)$$

$$= P(y, y'_z, x | z) + P(y, y'_z, x' | z)$$

Similarly to (9), we have

$$P(y, y'_z, x | z) - P(x, y, y'_z | z)$$

$$= P(y, y'_z, x | z) - P(y_z | z) + P(y, y'_z, x' | z)$$

$$= P(y | z) - P(y, y'_z, z) + P(x', y'_z, x | z)$$

Thus, the upper bounds are proved.

$$P(y_x, y'_z | z) - P(y'_z, x | z)$$

$$= P(y_x | z) - P(y'_z | z)$$

$$= P(y_x | z) - P(y'_z | z)$$

By (10),

$$z\text{-PNS} \geq P(y | z) - P(y, y'_z | z)$$

Also by (10) and (7),

$$z\text{-PNS} \geq P(y | z) - P(y, y'_z | z)$$

thus, the lower bounds are proved.

And since for any three events $A$, $B$ and $C$, we have

$$P(A, B | C) \leq \min[P(A | C), P(B | C)]$$

therefore, we have

$$z\text{-PNS} \leq \min[P(y_x | z), P(y'_z | z)]$$

Also, by (8),

$$z\text{-PNS} \leq P(x, y | z) + P(x', y'_z | z)$$

Similarly to (9), we have

$$z\text{-PNS} = P(y'_z | z) - P(y, y'_z, z | x) + P(x, y, y'_z | z)$$

$$= P(y'_z | z) - P(y, y'_z, z | x) + P(x, y, y'_z | z)$$

$$= P(y'_z | z) - P(y, y'_z, z | x) + P(x, y, y'_z | z)$$

Thus, the upper bounds are proved.
Lemma 7. The counterfactual expression \( f(\alpha) = \alpha P(y_x, y'_x | z) - (1 - \alpha) P(y_x, y'_x | z) \) for any real number \( \alpha \) are bounded as follows.

Case 1: \( \alpha \in (-\infty, 0.5) \)

\[
\begin{align*}
\max \left\{ \begin{array}{l}
\alpha P(y_x | z) - (1 - \alpha) P(y_x | z) \\
(1 - \alpha) P(y_x | z) + \alpha P(y'_x | z) + \alpha - 1 \\
(2\alpha - 1) P(y_x | z) + (2\alpha - 1) P(y'_x | z) + (1 - \alpha) P(y'_x, x'_x | z)
\end{array} \right. \\
\min \left\{ \begin{array}{l}
\alpha P(y_x | z) - P(y_x | z) \\
(1 - \alpha) P(y_x | z) - P(y_x | z) \\
(2\alpha - 1) P(y_g | z) + (1 - \alpha) P(y'_x | z) - \alpha P(y'_x | z)
\end{array} \right. \\
\geq f(\alpha)
\end{align*}
\]

Case 2: \( \alpha \in [0.5, \infty) \)

\[
\begin{align*}
\max \left\{ \begin{array}{l}
(1 - \alpha) [P(y_x | z) - P(y_x | z)] \\
\alpha [P(y_x | z) - P(y_x | z)] \\
(2\alpha - 1) P(y_g | z) + (1 - \alpha) P(y'_x | z) - \alpha P(y'_x | z)
\end{array} \right. \\
\min \left\{ \begin{array}{l}
\alpha P(y_x | z) - (1 - \alpha) P(y_x | z) \\
(1 - \alpha) P(y_x | z) + \alpha P(y'_x | z) + \alpha - 1 \\
(2\alpha - 1) P(y_g | z) + (2\alpha - 1) P(y'_x | z) + (1 - \alpha) P(y'_x, x'_x | z)
\end{array} \right. \\
\geq f(\alpha)
\end{align*}
\]

Proof. By lemma 6,

\[
\begin{align*}
f(\alpha) &= \alpha P(y_x, y'_x | z) - (1 - \alpha) P(y_x, y'_x | z) \\
&= \alpha P(y_x, y'_x | z) - (1 - \alpha) P(y_x, y'_x | z) - (1 - \alpha) P(y_x | z) + P(y_x | z)
\end{align*}
\]

By lemma 5, substituting (5) and (6) into (17), case 1 and 2 in lemma 7 hold.

Now, let’s prove theorem 1.

Proof.

\[
\begin{align*}
f(\beta, \gamma, \theta, \delta) &= \beta P(y_x, y'_x | z) + \gamma P(y_x, y'_x | z) + \theta P(y'_x, y'_x | z) + \delta P(y'_x, y'_x | z) \\
&= \beta P(y_x, y'_x | z) + \gamma [P(y_x | z) - P(y_x, y'_x | z)] + \theta [P(y'_x) - P(y_x, y'_x | z)] + \delta [P(y'_x) - P(y'_x, y'_x | z)]
\end{align*}
\]

By lemma 7, let \( \alpha = \frac{\beta - \gamma - \theta}{\beta - \gamma - \theta - \delta} \), substituting (13) to (16) into (18), theorem 1 hold.

8.2 Proof of Theorem 4

Lemma 8. If \( Y \) is monotonic relative to \( X \), \( z \)-specific \( PNS = P(y_x, y'_x | z) \) is identifiable whenever the causal effects \( P(y_x | z) \) and \( P(y'_x | z) \) are identifiable:

\[
PNS = P(y_x, y'_x | z) \\
= P(y_x | z) - P(y'_x | z).
\]

Proof. Since \( y'_x \) and \( y'_x \), are complementary, so \( y_x \lor y'_x = \) true, therefore, we have

\[
y_x = y_x \land (y_x \lor y'_x) = (y_x \land y'_x) \lor (y_x \land y'_x)
\]

Similarly,

\[
y'_x = y_x \land (y_x \lor y'_x) = (y_x \land y_x) \lor (y_x \lor y'_x)
\]

Since monotonicity entails that \( y'_x \land y'_x = \) false. Substituting (20) into (19) yields

\[
y_x = y_x \lor (y_x \land y'_x)
\]

Thus, for any \( z \), we have,

\[
y_x \land z = (y_x \land z) \lor (y_x \land y'_x \land z)
\]

Taking the probability of (21) and using the disjointness of \( y'_x \) and \( y'_x \), we obtain

\[
P(y_x, z) = P(y_x, z) + P(y_x, y'_x, z)
\]
Therefore,

\[ P(y_x|z) = P(y_{x'}|z) + P(y_x, y_{x'}|z) \]

or

\[ P(y_x, y_{x'}|z) = P(y_x|z) - P(y_{x'}|z) \] (22)

Now, let’s prove Theorem 4.

Proof.

\[
\begin{align*}
  f(\beta, \gamma, \theta, \delta) \\
  &= \beta P(y_x, y_{x'}|z) + \gamma P(y_x, y_{x'}|z) + \theta P(y_{x'}, y_{x'}|z) + \delta P(y_x, y_{x'}|z) \\
  &= \beta [P(y_x|z) - P(y_x, y_{x'}|z)] + \gamma [P(y_{x'}|z) - P(y_x', y_x'|z)] + \theta [P(y_{x'}|z) - P(y_x', y_x'|z)] + \delta [P(y_x|z) - P(y_x, y_{x'}|z)] \\
  &= \beta P(y_x|z) + (\gamma - \beta) P(y_{x'}|z) + \theta P(y_{x'}|z) + (\beta + \delta - \gamma - \theta) P(y_x, y_{x'}|z)
\end{align*}
\]

Thus, with \( \beta + \delta = \gamma + \theta \), theorem 4 hold.

Also if monotonicity, we have,

\[ P(y_{x'}, y_{x'}|z) = 0 \] (23)

By lemma 8, substituting (23) and (22) into (18), theorem 4 holds. \qed