Graphoids Over Counterfactuals

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Abstract

Augmenting the graphoid axioms with three additional rules enables us to handle independencies among observed as well as counterfactual variables. The augmented set of axioms facilitates the derivation of testable implications and ignorability conditions whenever modeling assumptions are articulated in the language of counterfactuals.

1 Motivation

Consider the causal Markov chain $X \rightarrow Y \rightarrow Z$ which represents the structural equations:

\begin{align*}
y &= f(x, u_1) \\
z &= g(y, u_2)
\end{align*}

(1)

(2)

with $u_1$ and $u_2$ being omitted factors such that $X, u_1, u_2$ are mutually independent.

It is well known that, regardless of the functions $f$ and $g$, this model implies the conditional independence of $X$ and $Z$ given $Y$, written

$X \indep Z \mid Y$ (3)

This can be readily derived from the independence of $X, u_1,$ and $u_2$, and it also follows from the $d$-separation criterion, since $Y$ blocks all paths between $X$ to $Z$.

However, the causal chain can also be encoded in the language of counterfactuals by writing:

\begin{align*}
Y_x(u) &= f(x, u_1) \\
Z_{xy}(u) &= g(y, u_2) = Z_y(u)
\end{align*}

(4)

(5)

where $u$ stands for all omitted factors (in our case $u = \{u_1, u_2\}$) and $Y_x(u)$ stands for the value that $Y$ would take in unit $u$ had $X$ been $x$. Accordingly, the functional and independence assumptions embedded in the chain model translate into the following counterfactual
Equation (6) represents the missing arrow from $X$ to $Z$, while (7)–(8) convey the mutual independence of $X, u_1,$ and $u_2$.\footnote{Rules for translating graphical models to counterfactual notation are given in Pearl (2009, pp. 232–234), based on the structural semantics of counterfactuals. The rules represent the omitted factors affecting any variable, say $Y$, by the set of counterfactuals $Y_{pa(Y)}$, where $pa(Y)$ stands for the parents of $Y$ in the diagram.}

Assume now that we are given the three counterfactual statements (6)–(8) as a specification of some uncharted model; the question arises: Are these statements testable? In other words, is there a statistical test conducted on the observed variables $X, Y,$ and $Z$ that could prove the model wrong? On the one hand, none of the three defining conditions (6)–(8) is testable in isolation, because each invokes a counterfactual entity. On the other hand, the fact that the chain model of Eqs. (1)–(2) yields the conditional independence of Eq. (3) implies that the combination of all three counterfactual statements should yield a testable implication.

This paper concerns the derivation of testable conditions like Eq. (3) from counterfactual sentences like Eqs. (6)–(8). Whereas graphical models have the benefits of inferential tools such as $d$-separation (Pearl 1988; 2009, p. 335) for deriving their testable implications, counterfactual specifications must resort to the graphoid axioms\footnote{The graphoid axioms are axioms of conditional independence, first formulated by Dawid (1979) and Spohn (1980). Their connections to graph connectivity and to other notions of “information relevance” were established by Pearl and Paz (1987) and are described in detail in (Pearl 1988, pp. 78–133; 2009, p. 11).}, which, on their own, cannot reduce subscripted expressions like Eqs. (6)-(8) into a subscript-free expression like Eq. (3). To unveil the testable implications of counterfactual specifications, the graphoid axioms must be supplemented with additional inferential machinery.

We will first prove that Eq. (3) indeed follows from Eq. (6)–(8) and then tackle the general question of deriving testable sentences from any given collection of counterfactual statements of the conditional independence variety. To that end, we will augment the graphoid axioms with three auxiliary inference rules, which will enable us to remove subscripts from variables and, if feasible, derive sentences in which all variables are unsubscripted, that is, testable. These auxiliary rules will rely on the composition axiom (Pearl, 2009, p. 229)

\[ X_w = x \implies Y_{xw} = Y_w \]  

which was shown to be sound and complete relative to recursive models (Galles and Pearl, 1998; Halpern, 1998).\footnote{The axiom of “composition” was first stated in Holland (1986, p. 968). Its completeness rests on a few technical conditions such as uniqueness and effectiveness (Halpern, 1998).} In the special case of $W = \{\emptyset\}$ the axiom is known as consistency rule:

\[ X = x \implies Y_x = Y \]  

and is discussed in Robins (1986) and Pearl (2010).
2 Deriving Testables from Non-testables

In this section we will show that Eq. (3) can be derived from (6)–(8) with the help of (9).

We first note that substituting (6) into (8) yields

\[ Z_y \perp \perp (Y_x, X) \]  

which is a universally quantified formula, stating that for all \( z, y, y', x, x' \) in the respective domains of \( Z, Y, \) and \( X, \) the following independence condition holds:

\[ Z_y = z \perp (Y_x = y', X = x') \]  

(12)

We next note that, for the special case of \( x' = x, \) Eq. (12) yields:

\[ Z_y = z \perp (Y_x = y', X = x) \]

or, using (10)

\[ Z_y = z \perp (Y = y', X = x) \quad \text{for all } y, z, y', x \]  

(13)

This can be written succinctly as

\[ Z_y \perp (Y, X) \]  

(14)

Our next task is to remove the subscript from \( Z_y. \) This is done in two steps. First we apply the graphoid rule of “weak union” (Pearl, 2009, p. 11) to obtain:

\[ Z_y \perp (Y, X) \implies Z_y \perp X \mid Y \]  

(15)

Second, we explicate the components of (15) and write

\[ Z_y \perp (X, Y) \implies Z_y = z \perp X = x \mid Y = y' \]  

(16)

for all \( y, z, x, \) and \( y'. \) Again, for the special case of \( y' = y, \) Eq. (10) permits us to remove the subscript from \( Z_y \) and write

\[ Z = z \perp X = x \mid Y = y \quad \text{for all } x, y, z \]  

(17)

Finally, since the last independency holds for all \( x, y, \) and \( z, \) we can write it in succinct notation as

\[ Z \perp X \mid Y \]

which is subscript-free and coincides with the testable implication of Eq. (3).

To summarize, we have shown that the subscripts in Eq. (11) can be removed in two steps. First

\[ Z_y \perp (Y_x, X) \implies Z_y \perp (Y, X) \]  

(18)

and second,

\[ Z_y \perp (Y, X) \implies Z \perp X \mid Y \]  

(19)

Moreover, we see that (3) follows from (8) alone, and does not require the exogeneity assumption expressed in (7).
3 Augmented Graphoid Axioms

In this section we will identify three general rules that, when added to the graphoid axioms, will enable us to derive testable implications without referring back to the consistency axiom of Eq. (10). The three rules are as follows

Rule 1
\[ V \perp \perp (X_w, Y_{zw}, S) \mid R \Rightarrow V \perp \perp (X_w, Y_w, S) \mid R \] (20)

Rule 2
\[ V \perp \perp R \mid (X_w, Y_{zw}, S) \Rightarrow V \perp \perp R \mid (X_w, Y_w, S) \] (21)

Rule 3
\[ V \perp \perp (Y_{zw}, S) \mid (X_w, R) \Rightarrow V \perp \perp (Y_w, S) \mid (X_w, R) \] (22)

Rules 1 and 2 state that a subscript \( x \) can be removed from \( Y_{zw} \) whenever \( Y_{zw} \) stands in conjunction with \( X_w \), be it before or after the conditioning bar. In our example we had \( W = \{\emptyset\} \). Rule 3 states that a subscript \( x \) can be removed from \( Y_{zw} \) whenever \( X_w \) appears in the conditioning set. The symbols \( V, S, R \) in Eqs. (20)–(22) stand for any set of variables, observable as well as counterfactual.

The proof of these three rules follow the path that led to the derivation of Eq. (18) and (19).

For mnemonic purposes we can summarize these rule using the following shorthand:

Rule 1–2
\[ (X_w, Y_{zw}) \Rightarrow (X_w, Y_w) \] (23)

Rule 3
\[ (Y_{zw}|X_w) \Rightarrow (Y_w|X_w) \] (24)

4 Deriving Ignorability Relations

Unveiling testable implications is only one application of the augmented graphoid axioms in Section 3. Not less important is the ability of these axioms to justify ignorability relations which a researcher may need for deriving causal effect estimands.4

Consider the sentence \( Z_x \perp \perp (Y_z, X) \) which may be implied by a certain process, and assume we wish to estimate the causal effect of \( Z \) on \( Y \), \( P(Y_z = y) \) from non-experimental data. For this estimation to be unbiased, the conditional ignorability \( Z \perp \perp Y_z\mid W \) need to be assumed, where \( W \) is some set of observed covariates. Using Axiom (22) we can show that

4Reliance on the assumptions of conditional ignorability (Rubin, 1974; Rosenbaum and Rubin, 1983; Holland, 1986), which are cognitively formidable, is one of the major weaknesses of the potential outcome framework (Pearl, 2009, pp. 350–351). Axioms (20)–(22) permit us to derive needed ignorability conditions from other counterfactual statements which are perhaps more transparent.
$W = X$ satisfies the ignorability assumptions and, therefore, adjustment for $X$ will yield a bias-free estimate of the causal effect $P(Y_z = y)$. This can be shown as follows:

$$Z_x \perp \perp (Y_z, X) \implies Z_x \perp \perp Y_z | X$$

(using the graphoid rule of “weak union”) and by Rule (22) we obtain

$$Z_x \perp \perp Y_z | X \implies Z \perp \perp Y_z | X$$

We therefore can write

$$P(Y_z = y) = \sum_x P(Y_z = y | X = x)P(X = x)$$

$$= \sum_x P(Y_z = y | Z = z, X = x)P(X = x)$$

$$= \sum_x P(Y = y | Z = z, X = x)P(X = x).$$

Equation (25) is none other but the standard adjustment formula for the causal effect of $Z$ on $Y$, controlling for $X$.

The process can also be reversed; we start with a needed, yet unsubstantiated ignorability condition, and we ask whether it can be derived from more fundamental conditions which are either explicit in the model or are defensible on scientific grounds. Consider, for example, an unconfounded mediation model in which treatment $X$ is randomized and assume we seek to estimate to effect of the mediator $Z$ on the outcome $Y$. (The model is depicted in Fig. 1). Operationally, we know that the ignorability condition $Z \perp \perp Y_z | X$ would allow us to obtain

![Figure 1: Unconfounded mediation model implying the conditional ignorability $Z \perp \perp Y_z | X$.](image)

the desired effect $P(Y_z = y)$ by adjusting for $X$, as shown in the derivation of Eq. (25). However, lacking graphs for guidance, it is not clear whether this condition follows from the assumptions embedded in the model; a formal proof is therefore needed. The assumptions explicit in the model take the form

(a) $X_z = X$

(b) $X \perp \perp (Z_x, Y_{zx})$

(c) $Z_x \perp \perp Y_{zx}$.

(a) states that $Z$ does not affect $X$, (b) represents the assumption that $X$ is randomized, and (c) stands for the no-confounding assumption, that is, all factors affecting $Z$ when $X$ is held
constant are independent of those affecting \( Y \) when \( X \) and \( Z \) are held constant (Pearl, 2009, p. 232, p. 343). These factors stand precisely for the “error terms” that enter the structural equations for \( Z \) and \( Y \), respectively, hence they have clear process-based interpretations, and avail themselves to plausibility judgments.

To show that the desired ignorability condition \( Z \perp \perp Y \mid X \) follows from (a), (b) and (c), we can use Rule 3 (Eq. 22) as follows. First, the standard graphoid axioms dictate

\[
X \perp \perp (Z_x, Y_{xx}) \& Z_x \perp \perp Y_{xx} \Rightarrow Z_x \perp \perp Y_{xx} \mid X
\]

Next, applying Rule 3 twice, together with \( X = X_z \), gives

\[
Z_x \perp \perp Y_{xx} \mid X \Rightarrow Z \perp \perp Y_{xx} \mid X \Rightarrow Z \perp \perp Y_{xx} \mid X_z \Rightarrow Z \perp \perp Y_z \mid X
\]

which yields the desired ignorability condition.

These derivations can be skipped, of course, when we have a graphical model for guidance. The adjustment formula (25) could then be written by inspection, since \( X \) satisfies the back-door condition relative to \( Z \rightarrow Y \). However, researchers who mistrust graphs and insist on doing the entire analysis by algebraic methods, would need to use Rules 1–3 to justify the ignorability condition from assumptions (a), (b), and (c).

5 Conclusions

Rules 1-3, when added to the graphoid axioms, allow us to process conditional-independence sentences involving counterfactuals and derive both their testable implications, as well as implications that are deemed necessary for identifying causal effects. We conjecture that Rules 1–3 are complete in the sense that all implications derivable from the graphoid axioms together with the consistency rule (18) are also derivable using the graphoid axioms together with Rules 1–3.

Augmented graphoids are by no means a substitute for causal diagrams, since the complexity of finding a derivation using graphoid axioms may be exponentially hard (Geiger, 1990). Diagrams, on the other hand, offer simple graphical criteria (e.g., \( d \)-separation or back-door) for deriving testable implications and effect estimands. In reasonably sized problems, these criteria can be verified by inspection, while, in large problems, they can be computed in polynomial time (Tian et al., 1998; Shpitser and Pearl, 2008). The secret of diagrams is that they embed all the graphoid axioms in their structure and, in effect, pre-compute all their ramifications and display them in graphical patterns.

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